Elementary Number Theory (TN410)

Exercises: Sheet #1

March 6, 2015

Volunteer students who, during problem sessions, will present the solution of one (or more) exercises (between number 4 and number 10 below), will get a bonus of one point on the final grade for each exercise solved at the blackboard.

1. Compute $\gcd(5520, 3135)$, $\gcd(8736, 3135)$;

2. Compute $v_2(70!)$, $v_5(125!)$ and $v_7(130!)$;

3. Let $a, b, c, n \in \mathbb{N}$. Show that

   (a) If $a \mid n$, $b \mid n$ and $\gcd(a, b) = 1$, then $ab \mid n$

   (b) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

4. Show that there exist infinitely many primes p of the form $p = 4k - 1$;
   (***hint:*** *Assume that* $p_1, \dots, p_k$ *are the only primes of this form and consider* $N = 4p_1 \cdots p_k - 1$)

5. Let $\pi(x) = \#\{p \le x\}$.

   (a) Compute (by hand or with a computer) $\pi(10)$, $\pi(100)$, $\pi(1000)$ and $\pi(10000)$;

   (b) Compare, in each case, the obtained value both with $X/\log X$ and with $\mathrm{li}(X)$.

6. Let, for $k > 1$, $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$. Show that

$$\sum_{n \le X} \frac{1}{n^k} = \zeta(k) + O\left(\frac{1}{X^{k-1}}\right) \qquad \text{and that} \qquad \sum_{n \le X} \frac{\mu(n)}{n^k} = \frac{1}{\zeta(k)} + O\left(\frac{1}{X^{k-1}}\right);$$

7. We say that $n \in \mathbb{N}$ is k–free if, for each prime p, $p^k \nmid n$. Let μ_k be the characteristic function of k–free integers. that is:

$$\mu_k(n) = \begin{cases} 1 & \text{if } n \text{ is } k\text{–free;} \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that μ_k is multiplicative;

   (b) Prove the identity:

$$\mu_k(n) = \sum_{\substack{d \in \mathbb{N} \\ d^k \mid n}} \mu(d);$$

   (c) Show that

$$\sum_{n \le X} \mu_k(n) = \frac{1}{\zeta(k)} X + O(X^{1/k}).$$

8. Show that the probability that two positive integers are coprime, is $6/\pi^2$;

9. Let N be an ipothetical odd perfect number. Show that the unique factorization of N has the form:

$$N = p_1^k \cdot p_2^{2j_2} \cdots p_r^{2j_r}$$

   where $k \ge 1$, $j_1, \dots, j_r \ge 1$ and $p_1 \equiv k \equiv 1 \bmod 4$;
   (***hint:*** *note that it must be* $\sigma(N) = 2N \equiv 2 \bmod 4$ *and deduce from it some properties of* $\sigma(p^\alpha)$ *for* $p^\alpha \| N$)

10. Show that an odd perfect number N cannot be of the form $6m - 1$.