Family Name .... *Name* .... *Student ID(Matricola):* ....

Solve the problems adding to the replies short and essential explenations. *Please write the solutions in the designed areas. NO EXTRA SHEETS WILL BE ACCEPTED.* 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

1	2	3	4	5	6	7	8	TOTAL

1. Compute $\gcd(1380, 1110)$ using the Extended Euclidean Algorithm and deduce a Bezout Identity.

2. Compute the 7–adic valuation $v_7(100!)$.

3. Let μ be the Möbius function and denote by $*$ the Dirichlet convolution of arithmetic functions. Prove that k–folded iterated convolution of μ satisfies:

$$(\mu * \mu * \cdots * \mu)(n) = \prod_p \binom{k}{v_p(n)} (-1)^{v_p(n)}$$

where for $a \in \mathbf{Z}$ and $b \in \mathbf{N}$, $\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}$ is the binomial coefficient.
(suggestion: try first to prove the formula for $k = 2, 3, ...$*)*

4. After having stated Gauss Theorem of existence of primitive roots modulo integers, compute all primitive roots modulo 686.

5. Find all integers X in the interval $[-10, 200]$ such that $\begin{cases} X \equiv 3 \bmod 4 \\ X \equiv 2 \bmod 5 \\ X \equiv 4 \bmod 7. \end{cases}$

6. After having stated the important properties of the Legendre–Jacobi Symbols, compute $\left(\frac{3073}{2919}\right)_J$.

7. Prove that

$$\left(\frac{-7}{p}\right)_{\mathrm{J}} = \begin{cases} 1 & \text{if } p \equiv 1, 2, 4 \bmod 7 \\ 0 & \text{if } p = 7 \\ -1 & \text{if } p \equiv 3, 5, 6 \bmod 7. \end{cases}$$

8. Let $p \geq 3$ be a prime and let $k \in \mathbf{N}$. Prove that

   i) the equation $X^k \equiv 1 \bmod p$ has $\gcd(k, p-1)$ solutions,

   ii) the equation $X^k \equiv 1 \bmod p^\alpha$ has $\gcd(k, p-1)$ solutions if $p \nmid k$.