Soluzioni 10-AM4

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1) Calcoliamo i coefficienti dispari

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\, dx \\ &= \frac{1}{\pi}\Big[e^x \sin kx\Big]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\, dx \\ &= -\frac{n}{\pi}\Big[e^x \cos kx\Big]_{-\pi}^{\pi} - \frac{k^2}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\, dx \end{aligned}$$

dunque

$$b_k = (-1)^{k+1}\frac{k}{k^2+1}\frac{2\sinh\pi}{\pi}.$$

Inoltre

$$c_0 = \frac{\sinh\pi}{\pi}.$$

Analogamente calcoliamo i coefficienti pari

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\, dx \\ &= \frac{1}{\pi}\Big[e^x \cos kx\Big]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\, dx \\ &= (-1)^k\left(2\frac{\sinh\pi}{\pi}\right) + (-1)^{k+1}\frac{k^2}{k^2+1}\left(2\frac{\sinh\pi}{\pi}\right) \end{aligned}$$

dunque

$$a_k = 2\frac{(-1)^k}{k^2+1}\frac{\sinh\pi}{\pi}.$$

Infine otteniamo

$$e^x = \frac{\sinh \pi}{\pi} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1} \frac{\sinh \pi}{\pi} \big[\cos kx - \sin kx \big].$$

2) I coefficienti di Fourier sono

$$\begin{aligned} b_k &= \frac{2}{\pi} \int_0^{\pi} e^x \sin kx \, dx \\ &= \frac{2}{\pi} \Big[e^x \sin kx \Big]_0^{\pi} - \frac{k}{\pi} \int_0^{\pi} e^x \cos kx \, dx \\ &= -\frac{k}{\pi} \Big[e^x \cos kx \Big]_0^{\pi} - \frac{k^2}{\pi} \int_0^{\pi} e^x \sin kx \, dx \end{aligned}$$

dunque

$$b_k = \frac{k}{k^2+1} \big[(-1)^k e^{\pi} - 1 \big].$$

Inoltre

$$c_0 = \frac{\sinh \pi}{\pi}.$$

Infine otteniamo

$$e^x = \frac{\sinh \pi}{\pi} + \sum_{k=1}^{\infty} \frac{k}{k^2+1} \big[(-1)^k e^{\pi} - 1 \big] \sin kx.$$

3) Sia $F(x) = x$. Essendo una funzione dispari basta calcolare i coefficienti della serie dei seni.

$$\begin{aligned} \hat{F}_k = b_k &= \frac{1}{\pi} \int_0^{2\pi} x \sin kx \, dx \\ &= \frac{1}{\pi} \Big[-\frac{x \cos kx}{k} \Big]_0^{2\pi} + \frac{1}{\pi} \int_0^{2\pi} \cos kx \, dx \\ &= -\frac{2}{k} \end{aligned}$$

Inoltre

$$\hat{F}_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx = \pi$$

dunque la serie di Fourier richiesta è

$$\begin{aligned} f(x) &= \frac{\pi - x}{2} \\ &= \frac{\pi}{2} + \frac{1}{2}\Big(-\pi + \sum_{k=1}^{\infty} \frac{2}{k} \sin kx \Big) \\ &= \sum_{k=1}^{\infty} \frac{\sin kx}{k} \end{aligned}$$

4) Basta osservare che

$$\begin{aligned} \sin^2 k^2 x &= \left(\frac{e^{ik^2x} - e^{-ik^2x}}{2i} \right)^2 \\ &= \frac{-e^{2ik^2x} - e^{-2ik^2x} + 2}{4} \\ &= \frac{1 - \cos 2k^2 x}{2} \end{aligned}$$

dunque la serie di Fourier associata ad f è

$$f(x) = \hat{f}_0 - \sum_{k=1}^{\infty} \frac{\cos 2k^2 x}{2^{k+1}}$$

dove

$$\hat{f}_0 = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}.$$