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ANALYTIC KAM TORI for the PLANETARY $(n+1)$-BODY PROBLEM

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Summary

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Abstract

In [7] Jacques Féjoz completed and gave the details of Michel Herman's proof of "Arnold's theorem" on the stability of planetary systems. This result provided the existence of C^∞ maximal invariant tori for the planetary $(n+1)$-body problem, with $n \geq 2$, in a neighborhood of Keplerian circular and coplanar movements, under the hypothesis that the masses of the planets are sufficiently small with respect to the mass of the "Sun". In this thesis we prove an analogous result in analytic class, i.e., we prove, under the same hypotheses listed above, the existence of real-analytic maximal invariant tori for the planetary $(n+1)$-body problem. The proof is based on the cited article by J.Fejoz and on a 2001 paper by H. Rüßmann. First, we prove a general quantitative theorem about existence of maximal KAM tori for nearly-integrable Hamiltonian systems near elliptic lower dimensional tori. Then, using [7], we obtain a set of initial data, in the phase space of the Hamiltonian model for a planetary system, with strictly positive Lebesgue measure, leading to quasi-periodic motions with $3n-1$ frequencies.

The thesis ends with three appendices. In appendices A and B we give a complete and detailed proof of Kolmogorov's original 1954 KAM theorem and we discuss a classical issue related to it and concerning the measure of invariant tori. In appendix C we briefly review Rüßmann's theory, contained in [16], about lower dimensional elliptic invariant tori for nearly-integrable Hamiltonian systems.

1 Introduction

The planetary $(n+1)$-body problem ($n \geq 2$) has always been one of the most relevant and discussed problem in the history of mathematics and physics. Nevertheless, it can be simply described by n planets and one star (considered as point masses) moving in the space under the effect of gravitational attraction. In 1963 it seemed that a theoretic solution to the stability of such motions had been finally found by V. I. Arnold. In [2] the Russian mathematician formulated a general result about existence of maximal invariant tori for the $(n+1)$-body problem in a neighborhood of Keplerian circular and coplanar movements. Actually, Arnold proved his statement only in the case of the planar three-body problem giving indications on how to generalize his approach to the general case; however, nobody has still succeeded in implementing Arnold's suggestions.

It is in [7] that a proof of Arnold's theorem appeared for the first time in his generality. In the cited article, J. Féjoz has completed and exposed Herman's work on the matter. In particular, in [7], the following result is established: there exists a positive measure set of phase space points, in the Hamiltonian model for the spatial planetary $(n+1)$-body problem, belonging to quasi-periodic motions with $3n-1$ frequencies and laying on C^∞ Lagrangian (maximal) tori [1]. In this thesis we prove the existence of such quasi-periodic motions in the analytic case.

Let m_0 be the mass of the "Sun", $m_1, \dots, m_n$ the masses of the n planets and $a_1 > a_2 > \cdots > a_n$ the semi major axes of the ellipses described by the planets. Then, if $0 < \epsilon < 1$ denotes the highest i^{th}-planet/Star mass ratio, the main result proved in this thesis can be formulated as follows:

Theorem 1 (Arnold's theorem on planetary motions in the real-analytic case). *For all values of masses $m_0, m_1, \dots, m_n$ and semi major axes $a_1 > a_2 > \cdots > a_n > 0$, there*

[1] Actually, in [2] V.I. Arnold announced a somewhat stronger result: "If the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion with suitable initial conditions throughout an infinite interval of time $-\infty < t < \infty$".

exists a real number $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, the flow of the spatial planetary Hamiltonian system possesses a strictly positive measure set of phase space points, in a neighborhood of circular and coplanar Keplerian tori with semi major axes $(a_1, a_2, \ldots, a_n)$, leading to quasi-periodic motions with $3n-1$ frequencies. Furthermore, such quasi-periodic motions lay on $(3n-1)$-dimensional real-analytic Lagrangian tori.

The proof of this theorem is based on two main results. The first is a theorem by H. Rüßmann, contained in [16], about existence of Lagrangian analytic KAM tori for nearly-integrable Hamiltonian systems [2]. A central role in Rüßmann's theorem is played by the non-degeneracy assumption made, which is exactly the non-degeneration in the sense of Rüßmann (definition 1) of the frequency application of the unperturbed Hamiltonian. Indeed, as it is well-known, the spatial planetary many-body problem is degenerate in the classical sense, i.e., it does not satisfy the non-degeneracy condition indicated by A.N. Kolmogorov in his 1954 theorem.

The other important result we are going to use is contained in the last chapter of [7]. There, J. Féjoz proves the non-degeneracy in the sense of Rüßmann of the planetary frequency application on an open and dense subset of the secular space having full Lebesgue measure[3].

In this summary, we first discuss briefly Rüßmann's theorem on maximal KAM invariant tori, underlining and explaining some slight differences between his original result and the one we will use for our purpose. Then, we consider a properly degenerate Hamiltonian function[4] in the same form of the planetary Hamiltonian expressed in Poincaré coordinates. Next, we perform some conformally symplectic transformations in order to remove the perturbation to a sufficiently high order and reduce our system to a degenerate case of that considered by Rüßmann. The scheme adopted is a classical one and is very similar to the scheme adopted by M. Herman in the first part of his general KAM theorem[5]. At this point we apply Rüßmann's theorem establishing a general KAM theorem for analytic properly degenerate Hamiltonians. Finally, using Fejoz and Herman's results we are able to conclude the proof of theorem 1.

2 Degenerate maximal tori theorem (après Rüßmann)

2.1 Rüßmann's theorem on analytic maximal KAM tori

Definition 1 (Rüßmann non-degeneracy condition). *Let B a non-void open connected set in $\mathbb{R}^n$, a real-analytic function $\omega = (\omega_1, \ldots, \omega_m) : B \longrightarrow \mathbb{R}^m$ is called* **non-degenerate** *if for any $(c_1, \ldots, c_m) \in \mathbb{R}^m \setminus \{0\}$*

$$c_1\omega_1 + \cdots + c_m\omega_m \neq 0$$

or equivalently if and only if the range $f(B)$ of f does not lie in any $(m-1)$-dimensional linear subspace of $\mathbb{R}^m$. We call ω **degenerate** *if it is not non-degenerate.*

As a simple consequence of analycity and non-degeneration, we are allowed to define the two following quantities:

[2] We remark that, in the cited paper, Rüßmann proves a much more general result concerning the existence of lower dimensional tori, both elliptic and hyperbolic; however in our thesis we use his result only in the special case of maximal tori (which is $p = q = 0$ with respect to Rüßmann's notation).

[3] This statement is not totally correct since the non-degeneracy of a slight modification of the planetary frequency application is proved. For detail see section 3.2.

[4] That is an Hamiltonian whose integrable part does not depend upon all the action variables.

[5] Refer to [9] or [7] for a complete proof of this very elegant and general theorem made in the C^∞ setting.

Definition 2. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set, $B \subseteq \mathbb{R}^n$ a domain containing $\mathcal{K}$ and $\omega : y \in B \longrightarrow \mathbb{R}^m$ a real-analytic and non-degenerate function. Let $\mathcal{S}^{n-1} := \{c \in \mathbb{R}^n \,:\, |c|_2 = 1\}$, we define $\mu_0(\omega, \mathcal{K}) \in \mathbb{N}_+$, the* **index of non-degeneracy** *of ω with respect to $\mathcal{K}$, as the first integer such that*

$$\beta := \min_{y \in \mathcal{K},\, c \in \mathcal{S}^{n-1}} \max_{0 \le \nu \le \mu_0} \left| D^\nu |\langle c, \omega(y) \rangle|^2 \right| > 0 \,; \tag{1}$$

$\beta = \beta(\omega, \mathcal{K})$ is called **amount of non-degeneracy** *of ω with respect to $\mathcal{K}$.*

We are now ready to formulate a qualitative version of Rüßmann's theorem

Theorem 2 (Rüßmann's theorem for maximal tori). *Let $\mathcal{Y}$ be an open connected set of $\mathbb{R}^n$ and $\mathbb{T}^n$ the usual n-dimensional torus $\mathbb{R}^n / 2\pi\mathbb{Z}^n$, consider a real-analytic Hamiltonian*

$$H(x, y) = h(y) + P(x, y)$$

defined for $(x, y) \in \mathbb{T}^n \times \mathcal{Y}$ endowed with the standard symplectic form $dy \wedge dx$.

Let $\mathcal{K}$ be any non-empty compact subset of $\mathcal{Y}$ with positive n-dimensional Lebesgue measure and fix $0 < \epsilon^\star <$ meas$_n\mathcal{K}$. Let $\mathcal{A}$ be an open set on which H can be holomorphically extended such that $\mathbb{T}^d \times \mathcal{K} \subset \mathcal{A}$.

Assume that the frequency application $\omega := \nabla h$ is non-degenerate in the sense of Rüßmann on $\mathcal{Y}$ and let $\bar{\mu}$ be any integer greater than index of non-degeneracy of ω with respect to $\mathcal{K}$.

Then, for any fixed $\tau > n\bar{\mu}$, there exist $\epsilon_0 > 0$ and $\gamma > 0$ depending on $\epsilon^\star, \mathcal{K}, \omega, \bar{\mu}, \beta, \mathcal{A}, \tau$ such that for $|P|_{\mathcal{A}} \le \epsilon_0$ the following is true: there exist a compact set $\mathcal{H} \subset \mathcal{K}$ with meas$_n\mathcal{H} >$ meas$_n\mathcal{K} - \epsilon^\star$ and a bi-lipschitz mapping

$$X : (b, \xi, \eta) \in \mathcal{H} \times \mathbb{T}^n \times \mathcal{U} \longrightarrow \mathbb{T}^n \times \mathcal{Y} \,,$$

where $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{R}^n$, such that

- *the mapping*

  $$(\xi, \eta) \longmapsto (x, y) = X(b, \xi, \eta)$$

  defines, for every $b \in \mathcal{H}$, a real-analytic canonical transformation on $\mathbb{T}^n \times \mathcal{U}$;
- *the transformed Hamiltonian $H^\star := H \circ X$ is in the form:*

  $$H^\star(b, \xi, \eta) = h^\star(b) + \langle \omega^\star(b), \eta \rangle + O(|\eta|^2)$$

  for every $b \in \mathcal{H}$ and $(\xi, \eta) \in \mathbb{T}^n \times \mathcal{U}$;
- *the new frequency vector $\omega^\star$ satisfies for all b in $\mathcal{H}$ the diophantine inequality*

  $$|\langle k, \omega^\star(b) \rangle| \ge \frac{\gamma}{|k|_2^\tau}, \;\; \forall\, k \in \mathbb{Z}^n \setminus \{0\} \,. \tag{2}$$

As we already remarked, this statement, concerning the existence of maximal tori only, is just a particular case of Rüßmann's main theorem contained in [16] [6].

However, there are two main differences between Rüßmann's theorem in the case of maximal invariant tori and theorem 2. The first consists in the different way of controlling the small

[6] We refer to the thesis to have some more details on how to obtain theorem 2 from the general results in [16]. In particular, in chapter 2 we review some important aspect of Rüßmann's paper only in the case of maximal invariant tori (the case $p = q = 0$ in Rüßmann's notations). Moreover, we give a quantitative formulation of theorem 2 which includes also an explicit estimate for the size of the perturbation.

denominators appearing in the problem. Indeed, Rüßmann performs this control in a very general way through a so called “approximation function”. Instead, our choice is to use classical Diophantine inequalities of the form (2). This apparently trivial aspect, which reduces also the measure of parameters leading to quasi-periodic motions, plays, on the contrary, a central role. In particular, we infer that it is necessary to consider such kind of inequalities to apply Rüßmann's theorem to the case of a properly degenerate Hamiltonian system adopting our scheme. Without going into details, we may only say that if we consider an approximation function as defined by Rüßmann in [16, definition 1.4] and then apply Rüßmann's estimate for the size of the perturbation to an Hamiltonian in the form (7), we get to a contradiction reaching an inequality of the form $O(\epsilon^N) \le O(\epsilon^\lambda)$ for any $\lambda > 0$, whereas $N \ge 1$ is a fixed real number [7].

The other important difference is the choice of $\bar{\mu}$ as any integer greater than the index of non-degeneracy of ω. In fact, we claim and prove that it is not necessary to use the literal definition of index of non-degeneracy, but the same results hold for any $\bar{\mu} \ge \mu_0$, if we take into consideration the corresponding “amount of non-degeneracy” defined by equation (1) with $\bar{\mu}$ instead of μ_0. The reason for this choice will be clarified later.

2.2 Formulation of degenerate maximal tori theorem

Let $\mathcal{B}$ an open set in $\mathbb{R}^d$, $\mathcal{U}$ some open neighborhood of the origin in $\mathbb{R}^{2p}$ and ϵ a “small” real parameter, we consider an Hamiltonian function H_ϵ in the form

$$H_\epsilon(\varphi, I, u, v) = h(I) + \epsilon f(\varphi, I, u, v) \tag{3}$$

and assume it is real-analytic for

$$(\varphi, I, (u, v)) \in \mathbb{T}^d \times \mathcal{B} \times \mathcal{U} =: \mathcal{M}$$

where $\mathcal{M}$ is endowed with the standard symplectic form

$$dI \wedge d\varphi + du \wedge dv\,.$$

Moreover, we assume that f is in the form $f(\varphi, I, u, v) = f_0(I, u, v) + f_1(\varphi, I, u, v)$ with

$$\int_{\mathbb{T}^d} f_1(\varphi, I, u, v)\, d\varphi = 0$$

and

$$f_0(I, u, v) = f_{00}(I) + \sum_{j=1}^{p} \Omega_j(I) \frac{u_j^2 + v_j^2}{2} + f_2(I, u, v) \tag{4}$$

with

$$\sup_{I \in \mathcal{B}} |f_2(I, u, v)| \le c_0 |(u, v)|^3\,, \quad \forall\ (u, v)\ \in \mathcal{U} \tag{5}$$

for some $c_0 > 0$.

Observe that the Hamiltonian $h + \epsilon f_0$ possesses for every $I_0 \in \mathcal{B}$ the invariant isotropic torus

$$\mathbb{T}^d_{I_0} := \mathbb{T}^d \times \{I_0\} \times \{0\} \subset \mathcal{M}$$

with corresponding quasi-periodic flow

$$\varphi(t) = \left[\frac{\partial h}{\partial I}(I_0) + \epsilon \frac{\partial f_{00}}{\partial I}(I_0)\right] t + \varphi_0 \qquad I(t) \equiv I_0 \qquad (u(t), v(t)) \equiv 0\,.$$

[7] See subsection 4.3.5 of the thesis for full details.

Disregarding the elliptic singularity in every single elliptic plane $u_j v_j$, we aim to find Lagrangian invariant tori for H_ϵ, i.e., maximal tori in the form

$$\mathbb{T}^{d+p}_{I_0,w} = \mathbb{T}^d \times \{I_0\} \times \{(u,v) \in \mathbb{R}^{2p} \ , \ |(u_j, v_j)| = w_j \ , \ \ \forall \ j = 1, \ldots, p\}$$

for I_0 in $\mathcal{B}$ and $w \in (\mathbb{R}_+)^p := \{w \in \mathbb{R}^p \ : \ w_j > 0 \ , \ \ \forall \ j = 1, \ldots, p\}$.

Theorem 3 (Degenerate maximal tori theorem). *Consider the real-analytic Hamiltonian function H_ϵ described above and assume that the "frequency map", i.e., the real-analytic application*

$$I \in \mathcal{B} \longrightarrow (\nabla h(I), \Omega_1(I), \ldots, \Omega_p(I)) \in \mathbb{R}^d \times \mathbb{R}^p \ , \tag{6}$$

is non-degenerate in the sense of Rüßmann (definition 1). Then, provided that ϵ is sufficiently small, in any neighborhood of $\mathbb{T}^d \times \{I_0\} \times \{0,0\} \subset \mathcal{M}$ there exists a positive measure set of phase space points belonging to real-analytic maximal KAM tori for H_ϵ carrying quasi-periodic motions.

The proof of the result described above consists of three main steps described by the following three theorems.

Theorem 4. *Assume that the frequency map*

$$I \in \mathcal{B} \longrightarrow (\nabla h(I), \Omega_1(I), \ldots, \Omega_p(I)) \in \mathbb{R}^d \times \mathbb{R}^p$$

is non-degenerate in the sense of Rüßmann. Chose and fix an integer $N \geq 2$ and an open set $V \subset \mathcal{B}$. Then, there exists an open ball

$$B^d(I_0, s) := \left\{I \in \mathbb{R}^d \ : \ |I - I_0| < s\right\} \subset V$$

such that, provided ϵ is small enough, there exists a real-analytic canonical transformation

$$\Phi_\epsilon : (\vartheta, r, \zeta, \rho) \ \in \mathbb{T}^d \times B^d(0, s/5) \times \mathbb{T}^p \times B^p(0, \epsilon) \longrightarrow$$

$$\longrightarrow (\varphi, I, u, v) \in \mathbb{T}^d \times B^d(I_0, s) \times \mathcal{U}$$

such that $\hat{H}_\epsilon := H_\epsilon \circ \Phi_\epsilon$ assumes the form

$$\hat{H}_\epsilon(\vartheta, \zeta, r, \rho) = \frac{1}{\epsilon} h(I_0 + \epsilon r) + \hat{g}(I_0 + \epsilon r) + \frac{1}{2}\hat{\Omega}(I_0 + \epsilon r) \cdot (\rho^0 + \epsilon\rho) +$$

$$+ Q_{\epsilon, I_0 + \epsilon r}(\rho^0 + \epsilon\rho) + \epsilon^N P_\epsilon(\vartheta, \zeta, r, \rho; \rho^0) \tag{7}$$

where ρ^0 in $(\mathbb{R}_+)^p$ is a chosen point having euclidean norm 2ϵ, $Q_{\epsilon, I_0+\epsilon r}$ is a polynomial of degree $N-1$ depending on ϵ and $I_0 + \epsilon r$, $\hat{g}$, $\hat{\Omega}$ and P_ϵ are real-analytic functions. Furthermore, if we denote $\Omega := (\Omega_1, \ldots, \Omega_p)$, it results

$$\sup_{r \in B(0,s/5)} |\hat{\Omega}(I_0 + \epsilon r) - \Omega(I_0 + \epsilon r)| = O(\epsilon) \, .$$

Theorem 5. *If ϵ is small enough, the frequency map of the torus $\mathbb{T}^{d+p}_{r,\rho}$ of the integrable part of $\hat{H}_\epsilon$, i.e., the real-analytic function*

$$\hat{\Psi}_\epsilon : (r, \rho) \in B^d(0, r/5) \times B^{2p}(0, \epsilon) \longrightarrow \left(\frac{\partial}{\partial r} F_\epsilon \frac{\partial}{\partial \rho} F_\epsilon\right)$$

where $F_\epsilon := \hat{H}_\epsilon - \epsilon^N P_\epsilon$, is non-degenerate in the sense of Rüßmann.

Moreover, let $\bar{\mu}$ and $\bar{\beta}$ denote respectively the index and amount of non-degeneracy of the initial frequency map in (6) with respect to a compact set $\overline{\mathcal{K}} \subset B^d(I_0, r/5)$; then, if we define $\hat{\mu}_\epsilon$ as the index of non-degeneracy of $\hat{\Psi}_\epsilon$ with respect to a suitable compact set $\mathcal{K} \subset B^d(0, r/5) \times B^p(0, \epsilon)$, and

$$\hat{\beta}_\epsilon := \min_{c \in \mathcal{S}^{d+p-1}} \min_{(r,\rho) \in \mathcal{K}} \max_{0 \le \nu \le \bar{\mu}} \left| \frac{\partial^\nu |\langle c, \hat{\Psi}_\epsilon \rangle|^2}{\partial r \partial \rho} \right| , \tag{8}$$

it results

$$\hat{\mu}_\epsilon \le \bar{\mu} \qquad \textit{and} \qquad \hat{\beta}_\epsilon \ge \frac{\epsilon^{\bar{\mu}+2} \bar{\beta}}{8} . \tag{9}$$

Theorem 6. *If N in theorem 4 is chosen to be sufficiently large and we assume that ϵ is small enough, then it is possible to apply Rüßmann's theorem for maximal KAM tori to $\hat{H}_\epsilon$ obtaining $\gamma > 0$ and a positive measure set of phase space points corresponding to quasi-periodic motions with (γ, τ)-Diophantine frequencies. Such motions lay on real-analytic maximal KAM tori.*

2.3 Proof of theorem 3

Before outlining a sketch of the proof of theorem 3, we introduce some notations:

- Let $A \subset \mathbb{R}^m$ or $\mathbb{C}^m$ and $|\cdot|$ denote the standard Euclidean norm; for any $t > 0$ we denote
  $$A_t := \bigcup_{x \in A} D^m(x, t) := \bigcup_{x \in A} \{x' \in \mathbb{C}^m \ : \ |x' - x| < t\}$$
  and
  $$\mathbb{T}^d_t := \{x \in \mathbb{C}^d \ : \ |\ \mathrm{Im}\ x_j| < t, \ \mathrm{Re}\ x_j \in \mathbb{T}\ , \ \forall\ j = 1 \dots d\} .$$
- We assume that H in (3) can be holomorphically extended to
  $$(\varphi, I, u, v) \in \mathbb{T}^d_\sigma \times \mathcal{B}_{r_0} \times \mathcal{U}_{r_1} =: \mathcal{M}_\star$$
  and denote
  $$\mu := \sum_{k \in \mathbb{Z}^d} \left(\sup_{\mathcal{B}_{r_0} \times \mathcal{U}_{r_1}} |f_k(I, u, v)| \right) e^{|k|_1 \sigma} .$$
  as the "sup-Fourier" norm of f. In particular, by the assumption made, H_ϵ is real-analytic on $\mathbb{T}^d \times B^d(I_0, s) \times B^{2p}(0, r_1)$ for any chosen I_0 in $\mathcal{B}$ and $s < r_0$.
- In this section we denote with C any positive constant which may depend on H, d, p, σ, r_0, r_1 and on s, as it will appear in inequalities (11) and (12).

We now follow the scheme described in section 2.2 and give a proof of theorems 4, 5 and 6 obtaining the validity of theorem 3 as a consequence.

The canonical transformation Φ_ϵ in theorem 4 is obtained through five main steps. First of all, consider the real-analytic Hamiltonian H_ϵ in (3) with the form described at the beginning of section 2.2, let $N_1 \ge 2$ and $N_2 \ge 3$ be two integers to be later determined and set

$$K_1 := \frac{6}{\sigma}(N_1 - 1) \log \frac{1}{\epsilon \mu} . \tag{10}$$

From the hypothesis of non-degeneracy in the sense of Rüßmann of the frequency application in (6), we may find $0 < s < r_0$ and a point $I_0 \in \mathcal{B}$ such that $B^d(I_0, s) \subset \mathcal{B}$ and the two following inequalities hold:

$$|\omega(I) \cdot k| > 0\,, \ \ \forall\, k \in \mathbb{Z}^d,\, 0 < |k|_1 \leq K_1, \quad \ \forall\, I \in D^d(I_0, s)\ , \tag{11}$$

$$|\Omega(I) \cdot k| > 0\,, \ \ \forall\, k \in \mathbb{Z}^p,\, 0 < |k|_1 \leq N_2, \quad \ \forall\, I \in D^d(I_0, s)\ . \tag{12}$$

Now, we apply a classical result of averaging theory in order to remove the dependence of H_ϵ on the "fast angles" φ up to order N_1. The following lemma is a corollary of a general formulation of "Averaging Theorem" given in [17, proposition A.1]:

Lemma 1 (Averaging Theorem). *In view of inequality (11) with K_1 defined in (10), for sufficiently small ϵ there exists a real-analytic symplectic transformation*

$$\Phi^1_\epsilon : (\tilde{\varphi}, \tilde{I}, \tilde{u}, \tilde{v}) \in \mathcal{M}_1 := \mathbb{T}^d_{\frac{\sigma}{6}} \times D^d(I_0, s/2) \times D^{2p}(0, r_1/2) \longrightarrow$$

$$\longrightarrow (\varphi, I, u, v) \in \mathcal{M}_0 := \mathbb{T}^d_\sigma \times D^d(I_0, s) \times D^{2p}(0, r_1)$$

that casts H_ϵ into the Hamiltonian

$$H^1_\epsilon := H_\epsilon \circ \Phi^1_\epsilon = h + g + \tilde{f}$$

where $g = g(\tilde{I}, \tilde{u}, \tilde{v})$ and $\tilde{f}$ satisfy

$$\sup_{\mathcal{M}_1} |g - \epsilon f_0| \leq C(\epsilon\mu)^2\ , \quad \sup_{\mathcal{M}_1} |\tilde{f}| \leq (\epsilon\mu)^{N_1}\ . \tag{13}$$

Now, in view of equation (13) we may write $g =: \epsilon f_0 + \tilde{g}$ with

$$\sup_{\mathcal{M}_1} |\tilde{g}| \leq C(\epsilon\mu)^2\ .$$

Thus, if we set $\tilde{g} =: \epsilon\bar{g}$ and $\tilde{f} =: \epsilon^{N_1}\bar{f}$, using again (13) we have

$$H^1_\epsilon(\tilde{\varphi}, \tilde{I}, \tilde{u}, \tilde{v}) = h(\tilde{I}) + \epsilon\left[f_0(\tilde{I}, \tilde{u}, \tilde{v}) + \epsilon\bar{g}(\tilde{I}, \tilde{u}, \tilde{v})\right] + \epsilon^{N_1}\bar{f}(\tilde{\varphi}, \tilde{I}, \tilde{u}, \tilde{v}) \tag{14}$$

with $\bar{g}$ and $\bar{f}$ real-analytic on $\mathcal{M}_1$. Furthermore, (4) and (5) give

$$f_0(\tilde{I}, \tilde{u}, \tilde{v}) = f_{00}(\tilde{I}) + \sum_{j=1}^{p} \Omega_j(\tilde{I}) \frac{\tilde{u}_j^2 + \tilde{v}_j^2}{2} + f_2(\tilde{I}, \tilde{u}, \tilde{v}) \tag{15}$$

with $|f_2(\tilde{I}, \tilde{u}, \tilde{v})| \leq C|(\tilde{u}, \tilde{v})|^3$ for every $(\tilde{u}, \tilde{v}) \in D^{2p}(0, r_1/2)$.

From equation (14) we see that the application of averaging theory has caused a shift of order ϵ to the elliptic equilibrium initially possessed by f_0 at the origin of $\mathbb{R}^{2p}$. We now focus our attention on the Hamiltonian function $f_0 + \epsilon\bar{g}$ with the aim to find a real-analytic symplectic transformation restoring the original equilibrium. The application of the Implicit Function Theorem yields the following: [8]

[8] Refer to section 3.3 in the thesis for detailed proofs of the following results.

Lemma 2. *Let $\mathcal{M}_2 := \mathbb{T}^d_{\frac{\sigma}{7}} \times D^d(I_0, s/4) \times D^{2p}(0, r_1/4)$; then, provided ϵ is small enough, there exists a real-analytic symplectic transformation*

$$\Phi^2_\epsilon : (x, y, p, q) \in \mathcal{M}_2 \longrightarrow (\tilde{\varphi}, \tilde{I}, \tilde{u}, \tilde{v}) \in \mathcal{M}_1$$

such that $H^2_\epsilon := H^1_\epsilon \circ \Phi^2_\epsilon$ is in the form

$$H^2_\epsilon(x, y, p, q) = h(y) + \epsilon \hat{g}(y, p, q) + \epsilon^{N_1} \hat{f}(x, y, p, q)$$

with $\partial_p \hat{g}(y, 0, 0) = 0 = \partial_q \hat{g}(y, 0, 0)$ and $\hat{g}$ and $\hat{f}$ real-analytic on $\mathcal{M}_2$.

Next, consider the quadratic part of $\hat{g}$, that is the real-analytic $2p \times 2p$ symmetric matrix $\hat{A}(y) := \partial^2_{(p,q)} \hat{g}(y, 0, 0)$. Using the construction of Φ^2_ϵ in lemma 2 and equation (15) for f_0, we obtain

$$\hat{A}(y) = \operatorname{diag}(\Omega_1(y), \ldots, \Omega_p(y), \Omega_1(y), \ldots, \Omega_p(y)) + O(\epsilon)\,.$$

Then, the following result runs as a consequence of another application of the Implicit Function Theorem:

Lemma 3. *If ϵ is taken sufficiently small, then the eigenvalues of the symplectic quadratic part of $\hat{g}$, i.e., the eigenvalues of*[9] *$\hat{A}(y) J_{2p}$, define $2p$ purely imaginary functions $\pm i\hat{\Omega}_1, \ldots, \pm i\hat{\Omega}_p$ verifying*

$$\sup_{y \in D^d(I_0, s/4)} |\hat{\Omega}(y) - \Omega(y)| \le C\epsilon\,. \tag{16}$$

Now, using a well-known result by K. Weierstraß on the symplectic diagonalization of quadratic Hamiltonians, we can find a real-analytic symplectic transformation $O(\epsilon)$-close to the identity

$$\Phi^3_\epsilon : (\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}) \in \mathcal{M}_3 := \mathbb{T}^d_{\frac{\sigma}{8}} \times D^d(I_0, s/5) \times D^{2p}(0, r_1/5) \longrightarrow (x, y, p, q) \in \mathcal{M}_2$$

such that $\tilde{y} = y$ and the transformed Hamiltonian function $H^3_\epsilon := H^2_\epsilon \circ \Phi^3_\epsilon$, which is real-analytic on $\mathcal{M}_3$, has the form

$$\begin{aligned} H^3_\epsilon(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}) &= h(\tilde{y}) + \epsilon \hat{g}_0(\tilde{y}) + \frac{\epsilon}{2} \sum_{j=1}^{p} \hat{\Omega}_j(\tilde{y}) \left(\tilde{p}_j^2 + \tilde{q}_j^2\right) + \\ &+ \epsilon \tilde{g}_3(\tilde{y}, \tilde{p}, \tilde{q}) + \epsilon^{N_1} \tilde{f}_3(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}) \end{aligned}$$

where $\hat{g}_0 := \hat{g}(\tilde{y}, 0, 0)$, $\tilde{f}_3 := \bar{f} \circ \Phi^3_\epsilon$ and $\tilde{g}_3 := \hat{g}_3 \circ \Phi^3_\epsilon$ verifies

$$\sup_{\tilde{y} \in D^d(I_0, s/5)} |\tilde{g}_3(\tilde{y}, \tilde{p}, \tilde{q})| \le C |(\tilde{p}, \tilde{q})|^3 \quad \forall\ (\tilde{p}, \tilde{q}) \in D^{2p}(0, r_1/5)\,.$$

Let $\tilde{g}_2(\tilde{y}, \tilde{p}, \tilde{q}) := \frac{1}{2} \sum_{i=1}^{p} \hat{\Omega}_i (\tilde{p}_i^2 + \tilde{q}_i^2)$, next step is putting $\tilde{g}_2 + \epsilon \tilde{g}_3$ into Birkhoff's normal form up to order N_2. In view of inequalities (12) and (16), provided ϵ is small enough, we have

$$|\hat{\Omega}(\tilde{y}) \cdot k| > 0\,,\ \ \forall\ k \in \mathbb{Z}^p,\ 0 < |k|_1 \le N_2,\ \ \forall\ I \in D^d(I_0, s/5)\,. \tag{17}$$

This non-resonance condition allows the following result:

[9] J_{2p} denotes the standard symplectic $2p \times 2p$ matrix.

Lemma 4 (Birkhoff's normal form). [10] *If inequality (17) is satisfied, then there exist* $0 < r_\star < r_1' \le r_1/5$ *and a real-analytic symplectic diffeomorphism*

$$\Phi_\epsilon^4 : (\theta, r, u, v) \in \mathcal{M}_4 := \mathbb{T}^d_{\frac{\sigma}{8}} \times D^p(I_0, s/5) \times D^{2p}(0, r_\star) \longrightarrow$$

$$\longrightarrow (\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}) \in \mathcal{M}_3' := \mathbb{T}^d_{\frac{\sigma}{8}} \times D^p(I_0, s/5) \times D^{2p}(0, r_1')$$

leaving the origin and the quadratic part of H_ϵ^3 *invariant, such that* $(\theta, r) = (\tilde{x}, \tilde{y})$ *and* $H_\epsilon^4 := H_\epsilon^3 \circ \Phi_\epsilon^4$ *is in the form*

$$\begin{aligned} H_\epsilon^4(\theta, r, u, v) &= h(r) + \epsilon \hat{g}_0(r) + \frac{\epsilon}{2} \sum_{j=1}^p \hat{\Omega}_j(r)(u_j^2 + v_j^2) + \\ &+ \epsilon Q_\star(r, u, v) + \epsilon R_\star(r, u, v) + \epsilon^{N_1} \tilde{f}_4(\theta, r, u, v) \end{aligned}$$

where:

- $Q_\star$ *is a polynomial of degree* $\left[\frac{N_2}{2}\right]$ *in the variables* $I = (I_1, \dots, I_p)$ *having the form*

  $$\langle \hat{\Omega}(r), I \rangle + \frac{1}{2} \langle T(r) I, I \rangle + \cdots \qquad \textit{with} \qquad I_j := \frac{1}{2}(u_j^2 + v_j^2)$$

  where $T(r)$ *is* $2p \times 2p$ *real-analytic matrix;*

- $R_\star$ *is a real-analytic function verifying* $|R_\star(r, u, v)| \le C |(u, v)|^{N_2+1}$ *for every* $(u, v) \in D^{2p}(0, r_\star)$ *and* $r \in D^d(I_0, s/5)$*;*
- $\tilde{f}_4 := \tilde{f}_3 \circ \Phi_\epsilon^4$ *is real-analytic on* $\mathcal{M}_4$*.*

Next, we perform a passage to symplectic polar coordinates in order to move $R_\star$ to the perturbative part of H_ϵ^4 with the help of a simple rescaling by a factor ϵ. Let $\rho^0 = (\rho_1^0, \dots, \rho_p^0)$ in $(\mathbb{R}_+)^p$ be sufficiently close to the origin; consider, for a suitable $\upsilon > 0$, the real-analytic symplectic transformation

$$\Phi_\epsilon^5 : (\theta, r, \zeta, \rho) \in \mathcal{M}_5 := \mathbb{T}^d_{\frac{\sigma}{8}} \times D^d(0, s/5) \times \mathbb{T}^p_\upsilon \times D^p\left(0, |\rho^0|/2\right) \longrightarrow$$

$$\longrightarrow (\theta, I_0 + r, z) \in \mathcal{M}_4$$

where $z = u + iv := \sqrt{\rho_j^0 + \rho_j}\, e^{-2i\zeta_j}$. So, the transformed Hamiltonian function $H_\epsilon^5 := H_\epsilon^4 \circ \Phi_\epsilon^5$, real-analytic on $\mathcal{M}_5$, assumes the form

$$\begin{aligned} H_\epsilon^5(\theta, r, \zeta, \rho) &= h(I_0 + r) + \epsilon \hat{g}_0(I_0 + r) + \frac{\epsilon}{2} \sum_{j=1}^p \hat{\Omega}_j(I_0 + r)(\rho_j^0 + \rho_j^0) + \\ &+ \epsilon Q_{I_0 + r}(\rho^0 + \rho) + \epsilon R(I_0 + r, \zeta, \rho^0 + \rho) + \epsilon^{N_1} \tilde{f}_5(\theta, r, \zeta, \rho; \rho^0) \end{aligned}$$

where

- $Q_{I_0+r} := Q_\star \circ \Phi_\epsilon^5$ is a polynomial of degree $\left[\frac{N_2}{2}\right]$ with respect to $\rho^0 + \rho$, depending also on $I_0 + r$;

[10] A complete and quantitative proof of "Birkhoff's normal form Theorem" can be found in section 3.4 of the thesis.

- $R := R_\star \circ \Phi_\epsilon^5$ verifies

$$|R(I_0 + r, \zeta, \rho^0 + \rho)| \leq C\left|\rho^0\right|^{\frac{N_2+1}{2}}$$

for every $\rho \in D^{2p}\left(0, \rho^0/2\right)$, $r \in D^d(0, s/5)$ and $\zeta \in \mathbb{T}_v^p$;

- $\tilde{f}_5 := \tilde{f}_4 \circ \Phi_\epsilon^5$ is real-analytic on $\mathcal{M}_5$.

Now, let A_ϵ be the homothety given by

$$A_\epsilon : (\theta, r, \zeta, \rho) \longrightarrow (\theta, \epsilon r, \zeta, \epsilon\rho)\,.$$

Even though A_ϵ is not a symplectic map, it preserves the structure of Hamilton's equations if we consider the Hamiltonian function $H_\epsilon^6 := \frac{1}{\epsilon} H_\epsilon^5 \circ A_\epsilon$. Explicitly we have

$$\begin{aligned} H_\epsilon^6(\theta, r, \zeta, \rho) &= \frac{1}{\epsilon} h(I_0 + \epsilon r) + \hat{g}_0(I_0 + \epsilon r) + \frac{1}{2}\hat{\Omega}(I_0 + \epsilon r) \cdot (\rho^0 + \epsilon\rho) \\ &+ Q_{\epsilon, I_0 + \epsilon r}(\rho^0 + \epsilon\rho) + R(I_0 + \epsilon r, \epsilon\rho, \zeta; \rho^0) + \epsilon^{N_1 - 1}\tilde{f}_6(\theta, r, \zeta, \rho; \rho^0)\,. \end{aligned}$$

where $\tilde{f}_6 := \tilde{f}_5 \circ A_\epsilon$. Now we fix $\rho^0 \in (\mathbb{R}_+)^p$ with $|\rho^0| = 2\epsilon$ such that we obtain $|R| \leq C\epsilon^{\frac{N_2+1}{2}}$. Thus, defining $N := N_1 - 1 := \frac{N_2+1}{2}$, we may write

$$R(I_0 + \epsilon r, \epsilon\rho, \zeta; \rho^0) + \epsilon^{N_1 - 1}\tilde{f}_6(\theta, r, \zeta, \rho) =: \epsilon^N P_\epsilon(\theta, r, \zeta, \rho)$$

for a suitable function P_ϵ real-analytic on $\mathbb{T}^d_{\frac{\sigma}{8}} \times D^d(0, s/5) \times \mathbb{T}_v^p \times D^p(0, \epsilon)$ (that is $\mathcal{M}_5$ once we have imposed $|\rho^0| = 2\epsilon$). We have so proved theorem 4 with $\hat{H}_\epsilon = H_\epsilon^6$.

Define now $F_\epsilon := \hat{H}_\epsilon - \epsilon^N P_\epsilon$ as the integrable part of $\hat{H}_\epsilon$, i.e.,

$$F_\epsilon(I_0 + \epsilon r, \rho^0 + \epsilon\rho) := \frac{1}{\epsilon} h(I_0 + \epsilon r) + \hat{g}_0(I_0 + \epsilon r) +$$

$$+\frac{1}{2}\sum_{j=1}^{p} \hat{\Omega}_j(I_0 + \epsilon r)(\rho_j^0 + \epsilon\rho_j) + Q_{\epsilon, I_0 + \epsilon r}(\rho^0 + \epsilon\rho)\,.$$

The frequency application of this unperturbed Hamiltonian is given by

$$\hat{\Psi}_\epsilon(r, \rho) := \left(\frac{\partial}{\partial r} F_\epsilon(I_0 + \epsilon r, \rho^0 + \epsilon\rho), \frac{\partial}{\partial \rho} F_\epsilon(I_0 + \epsilon r, \rho^0 + \epsilon\rho)\right) =$$

$$= \left(\omega(I_0 + \epsilon r) + O(\epsilon), \frac{\epsilon}{2}\hat{\Omega}(I_0 + \epsilon r) + O(\epsilon^2)\right)$$

and is real-analytic on $D^d(0, s/5) \times D^p(0, \epsilon)$ (we recall $\omega := \nabla h$). Then, using (16) we obtain

$$\hat{\Psi}_\epsilon(r, \rho) = \left(\omega(I_0 + \epsilon r) + O(\epsilon),\ \frac{\epsilon}{2}[\Omega(I_0 + \epsilon r) + O(\epsilon)]\right)\,. \tag{18}$$

Now, a proposition by Rüßmann gives the following characterization of non-degeneracy: if $f : B \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is real-analytic and non-degenerate, then for any point $b \in \mathcal{B}$ there exist m linearly independent coefficients in the Taylor expansion of f; conversely, if for some point $b \in B$ we can find such m linearly independent coefficients, then f is non-degenerate. This result establishes a relation between a non-degenerate function and the non-singularity of a matrix which depends on some of its derivatives. Thus, from hypothesis in theorem 3 and formula (18), we can conclude that $\hat{\Psi}_\epsilon$ is non-degenerate in the sense of Rüßmann on $D^d(0, s/5) \times D^p(0, \epsilon)$, provided that ϵ is small enough. The first part of theorem 5 is so proved.

Let now $\bar{\mu} \in \mathbb{N}_+$ and $\bar{\beta} > 0$ be respectively the index and the amount of non-degeneracy of Ψ with respect to a compact set $\overline{\mathcal{K}} \subset B^d(I_0, s/5)$ such that $\overline{\mathcal{K}} \ni I_0$. If we define $\mathcal{K}_0 := \left\{r \in \mathbb{R}^d \ : \ I_0 + r \in \overline{\mathcal{K}}\right\}$ and

$$\Psi_0(r) = (\omega(I_0 + \epsilon r), \Omega(I_0 + \epsilon r)) \ , \tag{19}$$

it results

$$\min_{r \in \mathcal{K}_0} \max_{0 \le \nu \le \bar{\mu}} \left| \frac{\partial^\nu}{\partial r} |\langle c, \Psi_0(r)\rangle|^2 \right| \ge \epsilon^{\bar{\mu}} \bar{\beta} > 0$$

for every $c \in S^{d+p-1} = \{c \in \mathbb{R}^{d+p} \ : \ |c|_2 = 1\}$. Now denote with Ψ_ϵ the real-analytic function over $D^d(0, s/5) \times D^p(0, \epsilon)$ obtained multiplying the last p component of $\hat{\Psi}_\epsilon$ by a factor $2/\epsilon$. Then, observe that equations (18) and (19) yield $\Psi_\epsilon(r, \rho) = \Psi_0(r) + O(\epsilon)$; thus, if ϵ is taken small enough, we may obtain

$$\min_{(r,\rho) \in \mathcal{K}_0 \times \mathcal{K}_1} \max_{0 \le \nu \le \bar{\mu}} \left| \frac{\partial^\nu}{\partial r} |\langle c, \Psi_\epsilon(r, \rho)\rangle|^2 \right| \ge \frac{\epsilon^{\bar{\mu}} \bar{\beta}}{2} > 0$$

for every $c \in S^{d+p-1}$, where $\mathcal{K}_1$ is some compact subset of $D^p(0, \epsilon/4)$ containing the origin. Furthermore, the definition of $\hat{\Psi}_\epsilon$ and Ψ_ϵ and the homogeneity of the function

$$f(c) := \min_{(r,\rho) \in \mathcal{K}_0 \times \mathcal{K}_1} \max_{0 \le \nu \le \bar{\mu}} \left| \frac{\partial^\nu}{\partial r} |\langle c, \Psi_\epsilon\rangle|^2 \right|$$

give

$$\min_{(r,\rho) \in \mathcal{K}_0 \times \mathcal{K}_1} \max_{0 \le \nu \le \bar{\mu}} \left| \frac{\partial^\nu |\langle c, \hat{\Psi}_\epsilon\rangle|^2}{\partial r \partial \rho} \right| \ge \frac{\epsilon^{\bar{\mu}+2} \bar{\beta}}{8} > 0$$

for every $c \in S^{d+p-1}$. Accordingly to the notation of theorem 5, if we take $\mathcal{K} = \mathcal{K}_0 \times \mathcal{K}_1$, the above inequality means $\hat{\mu}_\epsilon \le \bar{\mu}$. Moreover, $\hat{\beta}_\epsilon$, the "amount of non-degeneracy" [11] which corresponds to $\bar{\mu}$, verifies

$$\hat{\beta}_\epsilon \ge \frac{\epsilon^{\bar{\mu}+2} \bar{\beta}}{8} \tag{20}$$

such that theorem 5 has been completely proved.

Now, to end the proof of theorem 3 we need to apply Rüßmann's theorem for maximal tori to the "degenerate" case of $\hat{H}_\epsilon$. With theorems 4 and 5 we are in a position to meet the hypothesis of non-degeneracy of the frequency application required in theorem 2. However, the "degenerate" case of H^6_ϵ also requires that the size of the perturbation is of a sufficiently small order in ϵ.

As we see from (7), the size of the perturbation of $\hat{H}_\epsilon$ is order ϵ^N with $N \ge 2$; observe that since N_1 and N_2 can be arbitrarily fixed at the beginning of the process described in this section, also N can be arbitrarily *fixed*. Now, if we analyze the estimate given by Rüßmann[12], we see that the admissible size of the perturbation has a polynomial dependence upon the quantities $\epsilon^\star$, β, and $\mathcal{K}$, which, in turn, can be controlled by powers of ϵ when $\hat{H}_\epsilon$ is considered

[11] Notice that this is not literally the amount of non-degeneracy (unless $\bar{\mu} = \hat{\mu}_\epsilon$), but is the corresponding value of β in (1) with respect to $\bar{\mu}$.

[12] See [16, page 171] or the simplifi ed estimate that we derived in the maximal case in section 2.4 of the thesis.

($\hat{\beta}_\epsilon$ for instance is controlled by means of (20)). Moreover, accordingly to the statement of theorem 2, we can consider the index $\bar{\mu}$ in the application to $\hat{H}_\epsilon$ in view of $\bar{\mu} \geq \hat{\mu}_\epsilon$. Also, the exponent τ, which needs to be fixed in theorem 2, can be fixed a priori without any problem since the condition is $\tau > n\bar{\mu}$ which is greater than $n\hat{\mu}_\epsilon$. These observations explain how the admissible size of the perturbation of $\hat{H}_\epsilon$ is order $O(\epsilon^{N_0})$ for some $N_0 \geq 2$. Therefore, if we choose $N > N_0$ in theorem 4, we can apply Rüßmann's theorem to $\hat{H}_\epsilon$ and obtain theorem 6 as a consequence.

A fully detailed proof of the application of Rüßmann's theorem to $\hat{H}_\epsilon = H^6_\epsilon$ is provided in section 4.3 of the thesis. There, we analyze each quantity involved in the estimate given by Rüßmann for the admissible size of the perturbation, and in particular how they change order in ϵ when $\hat{H}_\epsilon$ is considered. Moreover, we give an explicit determination of a suitable lower bound for N, we explain the different choice for the control of the small denominators and we also provide an explicit estimate for the size of the perturbation in theorem 3.

3 Application to the planetary $(n+1)$-body problem

3.1 Hamiltonian formulation and reduction to the form considered in Theorem 3

The movements of $n+1$ bodies (point masses) interacting only through gravitational attraction are ruled by Newton's equations

$$\ddot{u}^{(i)} = \sum_{\substack{0\leq j\leq n \\ j\neq i}} \bar{m}_j \frac{u^{(j)} - u^{(i)}}{\left|u^{(i)} - u^{(j)}\right|^3}\ , \qquad i = 0, \dots, n \tag{21}$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in \mathbb{R}^3$ are the cartesian coordinates of the i^{th}-body of mass $\bar{m}_i$ and the gravitational constant has been renormalized to one by rescaling the time t.

As it is well know, the integral curves of equations (21) are the integral curves of the Hamiltonian vector field generated by the Hamiltonian function

$$\widetilde{H}_{\text{New}} := \sum_{i=0}^{n} \frac{\left|U^{(i)}\right|^2}{2\bar{m}_i} - \sum_{0\leq i<j\leq n} \frac{\bar{m}_i \bar{m}_j}{|u^{(i)} - u^{(j)}|}$$

where $U^{(i)} = \bar{m}_i u^{(i)}$ is the momentum conjugated to $u^{(i)}$, $(U^{(i)}, u^{(i)})$ are standard symplectic variables and the phase space considered excludes any intersection between the orbits.

The classical scheme adopted when dealing with many-body problems consists, at first, in the introduction of "heliocentric" coordinates and in a simple rescaling of masses by a factor ϵ (motivated by the planetary case in which one mass is much bigger than the others). Then, one introduces Poincaré coordinates $(\lambda, \Lambda, \xi, \eta, p, q) \in \mathbb{T}^n \times (0,\infty)^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. In this variables, the new Hamiltonian function, that we will call F_{plt}, has an integrable part given by

$$F_{\text{Kep}} := \sum_{1\leq j\leq n} -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2}$$

where μ_j and M_j are functions of $\bar{m}_0$, $\bar{m}_j$ and ϵ; the perturbative part of F_{plt} will be denoted by F_{per}. We can now define the *average movements* $\nu_1, \dots, \nu_n$, which play the role of ω_j in theorem 3[13] for $j = 1, \ldots, n$, by[14]

$$\nu_j := \frac{\partial F_{\rm Kep}}{\partial \Lambda_j} = \frac{\sqrt{M_j}}{a_j^{\frac{3}{2}}} = \frac{\mu_j^3 M_j^2}{\Lambda^3}\,. \tag{22}$$

Our aim is to apply theorem 3 with $H_\epsilon = H_{\rm plt}$, $f_0 = \langle F_{\rm per}\rangle$ [15] and $f_1 = F_{\rm per} - \langle F_{\rm per}\rangle$. Remarkable results by Laplace and Lagrange (contained for instance in [15]) about Birkhoff's normal form of $\langle F_{\rm per}\rangle$ near the elliptic equilibrium point at the origin, show the following fact: the quadratic part of $\langle F_{\rm per}\rangle$ can be written in the form

$$\sum_{1\le j\le n} \sigma_j(\xi_j^2+\eta_j^2) + \sum_{1\le j\le n} \varsigma_j(p_j^2+q_j^2) + O(4)\,.$$

where $\sigma_1, \ldots, \sigma_n$ and $\varsigma_1, \ldots, \varsigma_n$ are the eigenvalues of two bilinear forms denoted by $\mathcal{D}_h$ and $\mathcal{D}_v$, depending on the masses and semi-major axes via the famous "Laplace's coefficients". The Hamiltonian function of the planetary $(n+1)$-body problem is so reduced to the form considered in theorem 3 but we still need to check if it verifies the non-degeneracy hypothesis.

3.2 Laskar-Herman-Féjoz lemma about non-degeneracy

Up to rearranging the planets, we may assume that the semi major axes of the orbits described by the planets belong to the open subset of $\mathbb{R}^n$

$$\mathcal{A} := \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \,:\, 0 < a_n < a_{n-1} < \cdots < a_1\}\,.$$

Now, let ν_j be the average movements in (22), σ_j and ς_j the eigenvalues of the matrices representing $\mathcal{D}_h$ and $\mathcal{D}_v$, we denote with α the multivalued application

$$\alpha : a \in \mathcal{A} \longmapsto \{\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_n, \nu_1, \ldots, \nu_n\} \subset \mathbb{R}^n$$

and call it the **planetary frequency application**.

In [7, pages 52-62] (to which we always refer for full details) it is shown that, for all values of masses and in a simply connected neighborhood of almost every value of semi major axes, there exists an analytic determination of the frequency application again denoted by

$$\alpha : a \in \mathcal{A} \longmapsto (\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_n, \nu_1, \ldots, \nu_n) \subset \mathbb{R}^{3n}\,.$$

However, it turns out that this application is degenerate in the sense of Rüßmann. In particular the following statement is proved:

Proposition 1. *For all $n \ge 2$ there exists an open and dense set with full Lebesgue measure $U \subset \mathcal{A}$, where the eigenvalues of $\mathcal{D}_h$ and $\mathcal{D}_v$ are pairwise distinct and satisfy the following property: for any open and simply connected set $V \subset U$, the eigenvalues of $\mathcal{D}_h$ and $\mathcal{D}_v$ define $2n$ holomorphic functions $\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_n : V \longrightarrow \mathbb{C}$ which, together with the average movements $\nu_1, \ldots, \nu_n$, satisfy only this linear relations:*

$$\sum_{j=1}^n (\sigma_j + \varsigma_j) = 0 \qquad \textit{and} \qquad \varsigma_n = 0\,. \tag{23}$$

[13] With respect to notations in theorem 3 we have $d = n$ and $p = 2n$.
[14] We are using $\Lambda_j = \mu_j \sqrt{M_j\, a_j}$ by definition.
[15] This denotes the average of $F_{\rm per}$ with respect to $\lambda_1, \ldots, \lambda_n$ and is known as the "Secular Hamiltonian".

So, we are not yet in a position to apply directly theorem 3 to H_{plt}. To avoid the degeneracy of the frequency application, in [7] the author follows an intermediate strategy between a classical expedient (known as "Poincaré trick") and an idea of M. Herman. First of all, it is considered an Hamiltonian $H_\delta := H_{\text{plt}} - \delta C_z^2$, where δ is a real parameter and C_z is the third component of the total angular momentum. This new Hamiltonian H_δ still has an equilibrium point at the origin and its quadratic part possesses $4n$ eigenvalues with double multiplicity corresponding to $2n$ frequencies which form, together with the average movements, an "extended frequency application" $(\delta, a) \longmapsto \widetilde{\alpha}(\delta, a)$.

It is not difficult to prove that $\widetilde{\alpha}$ does not satisfy the first relation in (23). Furthermore, if the averaged Hamiltonian $\langle H_\delta \rangle$ is restricted to the submanifold of vertical total angular momentum $\mathcal{M}_{\text{vert}} := \{C_x = 0 = C_y\}$ and $\hat{\mathcal{D}}_\delta$ denotes its quadratic part computed at the origin, then the following is true

Theorem 7 (Herman, Féjoz, Laskar). *For all $n \geq 2$ there exists an open and dense set $U \subset \mathcal{A} \times \mathbb{R}$ with full Lebesgue measure, such that the $2n-1$ frequencies associated to the eigenvalues of $\hat{\mathcal{D}}_\delta$, regarded as functions of $a \in \mathcal{A}$ and $\delta \in \mathbb{R}$, are pairwise distinct and satisfy the following property: for every open and simply connected $V \subset U$ these frequencies define $2n-1$ holomorphic functions $\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_{n-1} : V \longrightarrow \mathbb{C}$ which, together with the average movements $\nu_1, \ldots, \nu_n$, do not satisfy any linear relation. In particular the frequency application*

$$\hat{\alpha} : (\Lambda, \delta) \in (\mathbb{R}_+)^n \times \mathbb{R} \to (\sigma_1, \ldots, \sigma_n, \nu_1, \ldots, \nu_n, \varsigma_1, \ldots, \varsigma_{n-1}) \in \mathbb{R}^{3n-1} \tag{24}$$

is non-degenerate in the sense of Rüßmann on an open and dense subset of $\mathcal{A} \times \mathbb{R}$ having full Lebesgue measure.

Now, a simple extension of theorem 3 to the case of an Hamiltonian depending on an additional parameter, permits its application to H_δ. Then, since H_{plt} and H_δ commute, a classical argument shows that they have the same invariant Lagrangian tori. So, we have obtained that H_{plt} possesses a positive measure subset of points in $\mathcal{M}_{\text{vert}}$ belonging to quasi-periodic motions, laying on real-analytic maximal invariant tori with $3n-1$ frequencies. Moreover, the union of these tori forms a strictly positive $(6n-2)$-dimensional Lebesgue measure set. Then, the invariance of equations (21) under rotations and Fubini's theorem show that the union of invariant tori for the planetary problem has strictly positive $6n$-dimensional Lebesgue measure.

4 Appendices

The thesis ends with three appendices in which we review some aspects of both classical and recent KAM theory. In appendix A we provide a proof of Kolmogorov's 1954 theorem on the persistence of quasi-periodic motion. Based on Kolmogorov's original and outstanding idea contained in [10], and inspired by notes taken from [5], we give a quantitative formulation of the famous theorem which marked the beginning of KAM theory. We underline the constructive aspect of the detailed proof by providing also an explicit estimate for the admissible size of the perturbation.

Next, in appendix B, we discuss another classical result concerning the measure of the union of Kolmogorov's tori (i.e., maximal invariant tori carrying quasi-periodic motions) which can be found with his theorem contained in appendix A. In particular, if we denote with ϵ the size of the perturbation of the considered Hamiltonian system, using C^∞-Whitney's extensions and the estimates obtained in Kolmogorov's theorem, we are able to conclude that the set of all invariant tori leaves out a set whose measure is proportional to $\epsilon^{\frac{1}{4}}$ times the measure of the phase space.

Finally, in appendix C, we briefly review the general Rüßmann's theory on invariant lower dimensional elliptic tori; we introduce some aspects of his theory and then summarize the main results contained in [16].

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