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Developments in non-integer bases: representability of real numbers and uniqueness

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Contents

Introduction

The subject of this thesis is the study of the power series developments of real numbers. We define a development of a real x any representation of x in the form

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i},$$

where coefficients are non negative integer digits of a finite alphabet $A = \{a_1, \ldots, a_m\}$. Theory about convergence of function series makes necessary the choice of a basis $\beta > 1$. Now, fixing an alphabet and a basis, we are interested to questions related to the representability of real numbers, the number of different expansions of representable numbers and the existence of unique expansions.

The classical approach consists in associating a development to an infinite sequence of digits belonging to the alphabet and to study these sequences. They are points of the space A^∞ and they are called *expansions*.

In its work "Representations for real numbers and their ergodic properties" (**[Rén57]**), A. Rényi proved some representability results. In particular he showed an algorithm which gives a standard expansion, the so-called greedy expansions, for every real $x \in [0, \frac{[\beta]}{\beta-1}]$ where the basis β is strictly greater than 1 and the alphabet is $A = \{0, 1, \ldots, [\beta]\}$. Greedy expansions have the property, which follows directly by their definition, to be the greatest possible expansions of the number which they represent with respect to the lexicographical ordering. This remark allowed Parry (see **[Par60]**) to characterize greedy expansions and to give a criterion to decide when a greedy expansion is the unique possible development of a given number.

Moreover, both Rényi's and Parry's articles discuss the problem of the entropy associated to the greedy expansions; in particular Rényi proved that the degree of disorder in the occurrence of digits is almost everywhere constant, thus it is almost everywhere independent from the represented number. Parry gave an explicit expression of this constant. These results are based upon the proof of the existence of a measure, Rényi's measure, which is invariant with respect to a transformation T such that its $n-th$ power, $T^n x$ associates to x the $n-$th rest of its greedy expansion. The existence of Rényi's measure, together with the ergodicity of T, makes possible applying ergodic theory to the problem of entropy of expansions and, later, it as been used by N. Sidorov (see **[Sid03]**) in order to prove that if $1 < \beta < 2$ then almost every representable number has a continuum of different expansions with alphabet $A = \{0, 1\}$.

We define as a gap the distance between two consecutive digits of an alphabet. Now, if representability with alphabets with constatly equal to 1 gaps, is granted with continuity on the whole interval $[\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}]$ for all $\beta > 1$, this is not true in the general case. In fact M. Pedicini proved in **[Ped05]** that the condition:

(1) $$\max_{1 \le i \le m-1} a_{i+1} - a_i \le \frac{a_m - a_1}{\beta - 1}$$

for all $a_i \in A = \{a_1, \ldots, a_m\}$, is necessary and sufficient to have that every $x \in I_\beta := [\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}]$ has a greedy expansion. Moreover, by notion of quasi greedy expansions Pedicini caracterized the greedy expansions and he showed a necessary and sufficient condition for an expansion to be unique. We show a generalization of Rényi's and Sidorov's results proving the existence of a generalized Rényi's measure and of a continuum of different expansions for almost every representable x with an alphabet A, which gaps satisfy condition 1. About uniqueness of expansions, we prove the existence of a constant $\bar{\beta}_A$ such that for all $\beta \leq \bar{\beta}_A$ there are not non-trivial unique expansions with basis β and in case $A = \{0, 1, 2^k\}$ such constant is equal to 2 for all $k \geq 1$.

CHAPTER 1

Preliminary Notions

We define the expansion of a real number as follows:

DEFINITION 1 (Expansions with respect to an alphabet A). *Let $A := \{a_1, \cdots, a_m\}$ a finite alphabet and $\beta > 1$.*

Given x a real number, an expansion of x with alphabet A is any representation of x in the form

$$x = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n} + \cdots$$

where $\epsilon_i \in A$ for all $i \geq 1$.

We now introduce the greedy expansion of a real number.

DEFINITION 2 (Greedy expansions). *Given any real x let us define* greedy expansion *the sequence $\epsilon_1, \epsilon_2, \ldots$ by the* greedy algorithm*: if ϵ_i is defined for any $i < n$, ϵ_n is the greatest digit in A satisfying*

$$\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \leq x$$

Following theorem, due to M. Pedicini (see [**Ped05**]), states a necessary and sufficien condition of representability for every $x \in I_\beta$.

THEOREM 1. *Let $A := \{a_1, \cdots a_m\}$ such that*

$$\max_{1 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_1}{\beta - 1}$$

then for every $x \in I_\beta$ we have

$$\sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x$$

where (ϵ_i) is given by the greedy algorithm.

Let us introduce the notion of quasi greedy expansion, which is usefull in order to characterize the greedy expansions.

DEFINITION 3. *Given any real x let us define the sequence $\epsilon_1, \epsilon_2, \cdots$ by the quasi-greedy algorithm: if ϵ_i is defined for any $i < n$, ϵ_n is the greatest digit in A satisfying*

$$\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} < x$$

Clearly an expansion of x is quasi-greedy and if and only if for all n

$$\sum_{i=1}^{n-1} \frac{\epsilon_i}{\beta^i} + \frac{\epsilon_n^+}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \geq x$$

Recall that $\epsilon_n^+ = a_{j_n+1}$, where $\epsilon_n = a_{j_n}$. Moreover a quasi-greedy expansion is always infinite.

REMARK 1. *Let us consider the lexicographical ordering: given two sequences (c_i) and (d_i) we write $(c_i) > (d_i)$ if $(c_i) \neq (d_i)$ and if $c_m > d_m$ for the first m such that $c_m \neq d_m$. By definition, we have that among all expansions of a given number $x \in I_\beta$ the greedy expansion is the biggest one, while the quasi-greedy expansion is the biggest among all infinite expansions. Clearly if the greedy expansion is infinite it coincides with the quasi-greedy one.*

Moreover, again by definition, we have that the map $x \to (\epsilon_i)$, where (ϵ_i) denotes the greedy expansion of x, is monotone.

Monotonicity of greedy expansions with respect to the basis is a property proved in the following proposition.

PROPOSITION 1. *Let x be a real number representable with two different bases $\beta < \overline{\beta}$ with respect to a common alphabet $A = \{a_1, \ldots, a_m\}$, that is $x \in I_\beta \cap I_{\overline{\beta}}$. Then the greedy (resp. lazy) expansion of x with basis β is lexicographically smaller than the greedy (resp. lazy) expansion of x with basis $\overline{\beta}$.*

Following notation allows us to introduce the Pedicini's characterization theorem of greedy expansions.

NOTATION 1. *Let us denote by $\epsilon'_i = \epsilon_i - a_1$, by $\Delta_i = a_{i+1} - a_i$, by $A' = \{a'_i = a_i - a_1 \mid i = 1, \cdots, m\}$ and by (γ^j_i) and (δ^j_i) respectively the greedy and quasi-greedy expansion of Δ_j with respect to the alphabet A'.*

THEOREM 2. *Let $A = \{a_i \mid i = 1, \cdots, m\}$ an alphabet of non-negative digits such that*

$$\max_{1 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_1}{\beta - 1}$$

then and expansion (ϵ_i) of $x \in I_\beta$ is greedy if and only if for all $n \geq 0$:

$$\epsilon'_{n+1}\epsilon'_{n+2} \cdots < \delta^{j_n}_1 \delta^{j_n}_2 \cdots$$

whenever $\epsilon_n < a_m$.

We conclude the chapter with a uniqueness result (see **[Ped05]**) which is a direct consequence of previous characterization theorem

NOTATION 2. *Given an alphabet $A = \{a_1, ..., a_m\}$ let us introduce the quasi-greedy expansion of differences:$\Delta_j = a_{j+1} - a_j = \frac{\delta^j_1}{\beta} + \frac{\delta^j_2}{\beta^2}$ for $j = 1, ..., m-1$.*

Let us also introduce the quasi-greedy expansion $(\bar{\delta}^j_i)$ of the difference $\bar{\Delta}_j = \bar{a}_{j+1} - \bar{a}_j = \frac{\bar{\delta}^j_1}{\beta} + \frac{\bar{\delta}^j_2}{\beta^2}$ with respect to the dual alphabet $\bar{A} = \bar{a}_1, ..., \bar{a}_m$ given by $\bar{a}_j = a_1 + a_m - a_{m+1-j}, j = 1, ..., m$..

Let us denote by $\mathcal{A}_q$ the set of numbers x whose greedy expansion (with respect to the original alphabet A) is the unique possible expansion.

THEOREM 3. *Assume that*

$$\max_{1 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_1}{\beta - 1}$$

Then for every $x \in I_\beta$ the greedy expansion of x, (ϵ_i),is unique if and only if

$$(\epsilon_{n+i} - a_1) < (\delta^j_i) \text{whenever } \epsilon_n = a_j < a_m, \text{ and}$$

$$(a_m - \epsilon_{n+i}) < (\bar{\delta}^j_i) \text{whenever } \epsilon_n = a_{m+1-j} > a_1.$$

CHAPTER 2

On the uniqueness of expansions

In previous chapter, we have seen that fixing an alphabet A and an adequate real number β, we have the existence at least of an expansion for all $x \in I_\beta$: the greedy one. Moreover, Theorem 3 shows a necessary and sufficient condition for the uniqueness of this expansion: an equivalent statement of Theorem 3, based upon the notion of lazy expansion, is exposed in 4.

DEFINITION 4. *A lazy expansion of x is obtained by the following algorithm:*

$$\lambda_n = \min\{a_i \in A | x \leq \sum_{i=1}^{n-1} \frac{\lambda_i}{\beta^i} + \frac{a_i}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i}\}$$

Clearly lazy expansion is the lexicographically smallest among the β expansions. Moreover the proof of the fact

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\beta^i} = x$$

is similar to the proof of convergence of greedy expansions.

NOTATION 3. *Let (δ_i^j) be the quasi-greedy expansion of j-th gap (i.e. $a_{j+1} - a_j$) in alphabet A. Let (λ_i^j) be the quasi-lazy expansion of $\frac{a_m}{\beta-1} - (a'_{j+1} - a'_j)$*

THEOREM 4. *Assume that*

$$\max_{1 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_1}{\beta - 1}.$$

Then for every $x \in I_\beta$ the greedy expansion of x, (ϵ_i),is unique if and only if

$$(\epsilon_{n+i} - a_1) < (\delta_i^j)$$

whenever $\epsilon_n = a_j < a_m$, and

$$(\epsilon_{n+i}) > (\lambda_i^j)$$

whenever $\epsilon_n = a_{m+1-j} > a_1$.

Observing hypothesis of Theorem 4 we can note that the existence of an unique expansion for some $x \in I_\beta$, fixing alphabet A, is conditioned to the choice of basis β. In fact if, for example, we for all $1 < j < m$ we have $(\lambda_i^j) > (\delta_i^j)$, hypothesis of Theorem 4 hold only in trivial cases $x = \frac{a_1}{\beta-1}$ and $x = \frac{a_m}{\beta-1}$. Now, following proposition state the existence of a "critical value" $\bar{\beta}_A$ which is the greatest basis which does not admit non-trivial unique expansions, where the trivial unique expansions of an alphabet $A = \{a_1, \ldots, a_m\}$ are $(a_1)^\infty$ and $(a_m)^\infty$.

PROPOSITION 2. *Let $A = \{a_1, ..., a_m\}$ an alphabet and $\beta' \in (1, \beta_A]$ such that the only unique expansions with basis β' are the trivial ones. Then for all $1 < \beta < \beta'$ we have that still the only unique expansions with basis β are the trivial ones.*

PROOF. Suppose that exists an expansion (c_j) such that satisfies conditions of Theorem 3 :

$$(c_{n+i} - a_1) < (\delta_i^j)$$

whenever $c_n = a_{j_n} < a_m$, and

$$(a_m - c_{n+i}) < (\bar{\delta}_i^j)$$

whenever $c_n = a_{m+1-j_n} > a_1$, where (δ_i^j) is the quasi greedy expansion of the gap $\Delta_j = a_{j+1} - a_j$ and $(\bar{\delta}_i^j)$ is the quasi greedy expansion of the gap $\bar{\Delta}_j = \bar{a}_{j_n+1} - \bar{a}_j$ where $c_n = a_j$and $\bar{a}_j \in D(A) = \{a_m + a_1 - a_{m+1-j} | j = 1, ..., m\}$, with respect to alphabet $D(A)$. Similarly we can define $(\delta_i^{\prime j})$ and $(\bar{\delta}_i^{\prime j})$ as the greedy expansions of gaps in alphabets A and $D(A)$ with basis β'. By monotonicity of quasi greedy expansions, (see Proposition 1) we have that

$$(\delta_i^j) < (\delta_i^{\prime j})$$

and

$$(\bar{\delta}_i^j) < (\bar{\delta}_i^{\prime j})$$

thus

$$(c_{n+i} - a_1) < (\delta_i^j) < (\delta_i^{\prime j})$$

whenever $c_n = a_j < a_m$, and

$$(a_m - c_{n+i}) < (\bar{\delta}_i^j) < (\bar{\delta}_i^{\prime j})$$

whenever $c_n = a_{m+1-j} > a_1$. Thus (c_j) is a non trivial unique expansion for β' and this an absurd because we have supposed that the only unique expansions with basis β' are the trivial ones, that is $(a_1)^\infty$ and $(a_m)^\infty$. Thus there are not unique expansions for any $\beta < \beta'$ and the proof is complete. □

NOTATION 4. *We denote as*

$$\overline{\beta}_A := sup_{1<\beta\leq\beta_A}\{\beta | \textit{there are not non trivial unique expansions with respect to } A\}$$

.

Next proposition states some invariance properties of $\bar{\beta}_A$ with respect some operations on the alphabet A.

PROPOSITION 3. *Let $A = \{a_1, ..., a_m\}$ and $\bar{\beta}_A$ its "critical value" then*

(1) $\bar{\beta}_{A+k} = \bar{\beta}_A$ where $k \in \mathbb{Z}$ and

$$A + k = \{a_i + k | a_i \in Ai = 1, ..., m\};$$

(2) $\bar{\beta}_{kA} = \bar{\beta}_A$ where $k \in \mathbb{Z}/\{0\}$ and

$$kA = \{ka_i | a_i \in Ai = 1, ..., m\};$$

(3) $\bar{\beta}_{D(A)} = \bar{\beta}_A$ where $D(A)$ is the dual of A, i.e.

$$D(A) = a_m + a_1 - a_{m+1-j} | a_i \in Ai = 1, ..., m.$$

PROOF. (1) Suppose that fixing $\beta < \bar{\beta}_A$ we have that

$$x \in I_\beta = \left[\frac{(a_1 + k)}{\beta - 1}, \frac{(a_m + k)}{\beta - 1}\right]$$

has two expansions (c_i) and (d_i) with respect alphabet $A + k$. This implies that $x - \frac{k}{\beta-1}$ has two expansions, with basis β with respect to alphabet A, given by $(c_i - k)$ and $(d_i - k)$ and this is a contradiction.

A	$\bar{\beta}_A$	Minimum Polynomial	Absolute Values of Conjugates
{0,1,2}	2.00000	-	-
{0,1,4}	2.00000	-	-
{0,1,8}	2.00000	-	-
{0,1,16}	2.00000	-	-
{0,1,32}	2.00000	-	-
{0,1,3}	2.18614	$2x^2 - 3x - 3$	-0.686141
{0,1,5}	2.11674	$-5 - 5x - x^2 - 5x^3 - 5x^4 -$ $x^5 - 5x^6 - 5x^7 + 4x^8$	0.83276; 0.904695; 0.96167; 0.967907
{0,1,6}	2.06341	$5x^3 - 6x^2 - 6x - 6 = 0$	0.762602
{0,1,7}	2.02647	$6x^3 - 7x^2 - 7x - 7 = 0$	0.758757
{0,1,9}	2.05889	$8x^7 - 9x^6 - 9x^5 - 9x^4 - x^3 - 9x^2 -$ $9x - 9 = 0$	0.889114; 0.85938; 0.967423

TABLE 1. Table of Minimun Polynomials

(2) As in previous point we can suppose the existence of (c_i) and (d_i) as β expansions for some $x \in \left[\frac{(ka_1)}{\beta-1}, \frac{(ka_m)}{\beta-1}\right]$ if $k > 0$ or $x \in \left[\frac{(ka_m)}{\beta-1}, \frac{(ka_1)}{\beta-1}\right]$ if $k < 0$ with respect to alphabet kA we have that $(\frac{c_i}{k})$ and $(\frac{d_i}{k})$ are two distinct β expansion for $\frac{x}{k} \in I_\beta$ with respect to alphabet A.

(3) By above proofs, it is sufficient to observe that the operation $D(A)$ is the combination of a translation (we add a_m to every digit of A) and a product for a constant $k = -1$.

□

By Proposition 3 we have that without loss of generality we can restrict our research of $\bar{\beta}_A$ to alphabets such that their first digit is zero, their digits are coprime (i.e. greatest common divisor of all digits is 1) and excepting operation dual. By applying the criterion exposed in Theorem 4, we get the sperimental datas reported in Table 1, by an algoritmh implemented with Mathematica.

By data reported above we can deduce that β_A are Pisot numbers for all examined alphabets: in fact we have that all the conjugates of β_A are in modulus lower than 1. Let us recall the definition of Pisot number:

DEFINITION 5 (Pisot Numbers). *A Pisot number is an algebraic number q which modulus is greater than 1 and exists a polynomial $P(x)$ such that $P(q) = 0$ (i.e. q is a root for $P(x)$) and all conjugates of q are in modulus strictly lower than 1*

EXAMPLE 1. *The golden mean $q = \frac{1+\sqrt{5}}{2}$ is a Pisot number, because it is clearly greater than 1 and its minimun polynomial is*

$$x^2 - x - 1 = 0$$

and its conjugate is $|\frac{1-\sqrt{5}}{2}| = |-0.618034| < 1$

Excepting the case $A = \{0, 1, 5\}$, greedy expansions reported in Table 1, and related to alphabets in the generic form $A = \{0, 1, m\}$, are in the form $(m)^k$ where k is such that $2^k \geq m$.

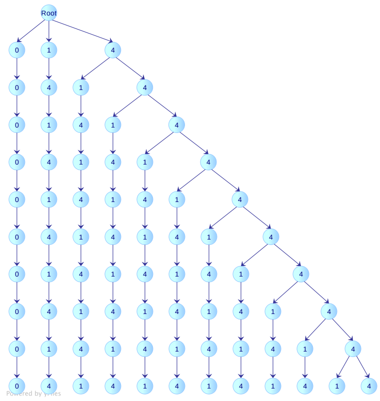

FIGURE 1. unique expansions with basis $2+\epsilon$ in case $A = \{0, 1, 2\}$

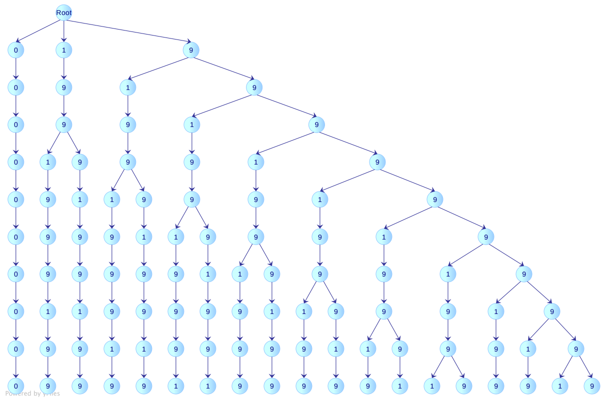

FIGURE 2. unique expansions with basis 2 in case $A = \{0, 1, 8\}$

This, together with the regularity of the structure of empirically detected unique expansions of alphabets $A = \{0, 1, 4\}$ and $A = \{0, 1, 9\}$, reported in Fig. 1 and Fig. 2, leads us to state a criterion to estabilish when a basis admits some unique expansions.

First of all we need the following lemmas:

LEMMA 1. *Let (c_i) be the greedy expansion of gap $\Delta_j = a_{j+1} - a_j$. If (c_i) is finite, that is there n $c_n \neq a_1$ and $c_n + i = a_1$ for all $i \geq 1$, and if $c_n = a_{j+1}$ then the quasi greedy expansion of Δ_j is $(c_1, \ldots, c_{n-1}, a_j)^\infty$.*

LEMMA 2. *For every $m \geq 3$ and for every k such that $m \leq 2^k$ the polynomial*

$$P(x) = (m-1)x^{k+1} - (2m-1)x^k + m$$

has one root in the interval $[2, m]$.

Following theorem states that $\bar{\beta}_A leq \bar{\beta}$ where $\bar{\beta}$ is a basis such that the greedy expansion of $m-1$ is in the form $(m)^k$, with an adequate k. Thus every $\beta > \bar{\beta}$ admit some unique expansion.

THEOREM 5. *Let $A = \{0, 1, m\}$ and $m \geq 3$. If the greedy expansion of $m-1$ is $(m)^k(0)^\infty$, where $2^k \geq m$ with respect to the basis $\bar{\beta}$ then, for all $\beta > \bar{\beta}$, expansions $c_n^p := m^p(1m^k)^\infty$ are unique with respect to β and A.*

PROOF. Let δ_i^1 and δ_i^2 (resp. γ_i^1 and γ_i^2) quasi greedy expansions of 1 and $m-1$ with respect to A (resp. dual alphabet $D(A)$). Now, for every $n \geq 1$ we have that:

(1) If $n \leq p$ then

$$c_n^p = m \text{and} (m - c_{n+i}^p) = 0^{p-n}(m - 10^k)^\infty < \gamma_i^2 \tag{2}$$

In fact we have that

$$\begin{aligned}(m - c_{n+i}^p)(\bar{\beta}) &= \frac{1}{\bar{\beta}^{p-n+1}} \frac{m-1}{\bar{\beta}^k - 1} \\ &\leq m - 1\end{aligned}$$

Because $2 \leq \bar{\beta}$. By monotonicity of quasi greedy expansions with respect the represented number we have that $(m - c_{n+i}^p)$ is lexicographically smaller than the quasi greedy expansion of $m - 1$ with respect to $D(A)$ and by monotonicity of quasi greedy expansions with respect to the basis we have $(m - c_{n+i}^p) < \gamma_i^2$

(2) If $n > p$ and $n \neq p + kj + 1$, let $\bar{n} := n - p(mod k)$

$$c_n^p = m \text{and} (m - c_{n+i}^p) = 0^{k-\bar{n}}(m - 10^k)^\infty < \gamma_i^2$$

The proof is similar to the one related to 2

(3) If $n = p + kj + 1$, we have

$$c_n^p = 1 \text{and} \quad c_{n+i} = (m^{k-1}1)^\infty < \delta_i^1 \quad m - c_{n+i} = (0^{k-1}m - 1)^\infty < \gamma_i^1 \tag{3}$$

Inequality (3) follows by monotonicity of quasi greedy expansions with respect to the basis and by the fact that, by Lemma 1 the quasi greedy expansion of $m - 1$ with basis $\bar{\beta}$ is equal to c_{n+i}.

Inequality (3) follows by the following consideration: $(m - m - c_{n+i})(\bar{\beta}) = \frac{m-1}{\bar{\beta}^k - 1} < m - 1$ because $\beta \geq 2$. Thus, again by monotonicity properties of quasi greedy expansions we have that (3) holds for all $m \geq 3$.

Thus for all $p \geq 0$ expansions (c_n^p) are unique for all bases $\beta > \bar{\beta}$. □

1. An example:case $A = \{0, 1, 2^k\}$

In this section we show that $\beta = 2$ is the greatest basis such that an unique expansion for any $x = I_\beta$ does not exist.

THEOREM 6. *$\bar{\beta}_A = 2$ is the "crical value" for alphabets $A_k = \{0, 1, 2^k\}$, for all $k > 1$.*

PROOF. We can observe that the greedy expansion of the greatest gap in A_k, $2^k - 1$ is $(2^k)^k$. In fact

$$\sum_{i=1}^{k} \frac{2^k}{2^i} = \sum_{i=0}^{k-1} 2^i = 2^k - 1$$

and $(2^k)^k$ is clearly the greatest expansion of $2^k - 1$ with respect to the lexicographical ordering. Thus $(2^k)^k$ is greedy. Applying Theorem 5 we have to prove that the choice of 2 as basis does not admit non-trivial unique expansions. Suppose that (d_n) is an unique expansion for with basis 2. First of all, by Theorem 4, a necessary condition for the occurrence of 1 in (d_n) is that $\lambda_i^1 < \delta_i^2$ but λ_i^1 δ_i^2 expand the same number $(2^k - 1)$ thus they are equal: this excludes the occurrence of 1 in (d_n). Let us prove that strings 2^k0 and 02^k cannot occur in (c_n), too. Let us observe that the greedy expansion of 1, the gap related to 0, is equal to $(1)^\infty$. In particular, since it is infinite it coincides with the quasi greedy expansion of 1. Now, if $c_n = 0$ and $c_{n+1} = 2^k$ then

$$(c_{n+i}) = 2^k(c_{n+1+i}) > (1)^\infty = \delta_i^1 i \geq 1. \tag{4}$$

Thus 02^k cannot occur in (d_n). Moreover inequality (4) implies that if $d_{n+1} = 0$, for some $n \geq 1$ then $(d_{n+1+i})_{i\geq 1} = (0)^\infty$. Now suppose that $c_n = 2^k$ and $c_{n+1} = 0$:

$$2^k - (c_{n+i}) = (2^k)^\infty \geq \gamma_i^2 i \geq 1.. \tag{5}$$

Hence the only unique expansions with basis $\beta = 2$ are the trivial ones. This completes the proof. □

CHAPTER 3

The number of different expansions

In this section we see that almost every real $x \in I_\beta$ has a continuum of β -expansions $\forall 1 < \beta < 2$ with respect to the alphabet $A = \{a_1, \dots, a_m\}$, with "small gaps", i.e. such that every gap $a_{j+1} - a_j < \frac{a_m - a_i}{\beta - 1}$ where $a_j \in A$ and $j = 1, \dots, m-1$.Note that at this moment we are sure of the existence the only greedy expansion for any real $x \in I_\beta$.

The arguments in this section are a generalization of Sidorov's and Rényi's results in case of $A = \{0, 1\}$(see **[Sid03]** and **[Rén57]**).

First of all we define a transformation T_β such that $T^n x$ for all $x \in I_\beta$ associates to x the rest of its greedy expansion of order n. Formally we have the following:

DEFINITION 6. *Let*

$$T(x) = \beta(x - \frac{\epsilon_1}{\beta})$$

where (ϵ_i) is the greedy expansion of x.

REMARK 2. *If $x \in I_\beta$ and (ϵ_i) is its greedy expansion, by definition of T we have that:*

$$x = \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \frac{T^n x}{\beta^n}$$

Now, we have defined greedy expansions as the result of an algorithm applied on $x \in I_\beta$, but in order to prove the existence of a continuum of expansion for almost every x we need to change point of view. Consider the dynamical system $(I_\beta, \mathcal{B}, \mu_\beta, T)$, where μ_β is the Lebesgue's measure, normalized on I_β. Now we consider the partition of I_β :

$$\mathcal{I} := I_i = \left[\frac{a_i}{\beta} + \sum_{i=1}^{\infty} \frac{a_1}{\beta^i}, \frac{a_{i+1}}{\beta} + \sum_{i=1}^{\infty} \frac{a_1}{\beta^i}\right) |i = 1, \dots, m-2$$

$$\cup I_m = \left[fraca_m\beta + \sum_{i=1}^{\infty} \frac{a_1}{\beta^i}, fraca_m\beta + \sum_{i=1}^{\infty} \frac{a_m}{\beta^i}\right]$$

and we see that we can equivalently define the greedy expansion of x as a sequence of digits in A such that $\epsilon_n = a_{i_n}$ if and only if $T^n x \in I_{i_n}$.In this way a greedy expansion is nothing more than a representation of the orbit of x in I_β under the action of transformation T. This allows us to use classical results of ergodic theory for our purposes.

Now, we define the so called *canonicalsequences*: they are finite sequences of digits of A and of a certain finite length n corresponding to the first n terms of a greedy expansion for some $x \in I_\beta$. Equivalently a sequence $(\epsilon_i)_{i=1}^n$ is canonical if exists $x \in I_\beta$ such that $T^i x \in I_{j_i}$ where $\epsilon_i = a_{j_i} \in A$ for all $i = 1, \dots, n$.

DEFINITION 7 (canonical sequences). *A sequence $(\epsilon_1)_{i \le n}$ is said to be canonical if there is a greedy expansion such that its first n terms are equal to $(\epsilon_i)_{i \le n}$.*

LEMMA 3. *Let $(\epsilon_1)_{i \le n}$ be a canonical sequence of order n. Then for all $a_i \in A$ such that $a_i \le \epsilon_n$ then $(\epsilon_1, \dots, \epsilon_{n-1}, a_i)$ is a canonical sequence.*

Following theorem is a generalization of Renyi's Theorem and states the existence of a measure ν_β invariant with respect T.

THEOREM 7. *Let $A = \{a_1, \cdots, a_m\}$ an alphabet such that*

$$\max_{1\leq j\leq m-1}(a_{j+1}-a_j) \leq \frac{a_m - a_1}{\beta - 1}$$

. Then for any function which is L-integrable on $\left(\frac{a_1}{\beta-1}, \frac{a_m}{\beta-1}\right)$ we have for almost all xin I_β

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g(T^k x) = M(g)$$

where $M(g)$ is a constant that does not depend on x. Moreover exists a measure ν equivalent to the Lebesgue measure μ and invariant with respect to T.

PROOF. Let $S(n)$ the number of canonical sequences of order n and put $S(0) = 1$.

So $S(n) - S(n-1)$ is the number of canonical sequences of order n such that $\epsilon_n \neq a_1$. In fact, if $(\epsilon_1, \cdots, \epsilon_{n-1})$ is a canonical sequence then $(\epsilon_1, \cdots, \epsilon_{n-1}, a_1)$ is it, too; and clearly if $(\epsilon_1, \cdots, \epsilon_n)$ is canonical then $(\epsilon_1, \cdots, \epsilon_{n-1})$ is canonical. Now let k_ξ such that $(\epsilon_1, \cdots, \epsilon_{n-1}, a_k)$ is canonical if and only if $1 < k \leq k_\xi$: intervals

$$\left[\sum_{i=1}^{n-1}\frac{\epsilon_i}{\beta^i} + \sum_{i=n}^{\infty}\frac{a_1}{\beta^i}, \sum_{i=1}^{n-1}\frac{\epsilon_i}{\beta^i} + \frac{a_{k_\xi}}{\beta^n} + \sum_{i=n+1}^{\infty}\frac{a_1}{\beta^i}\right]$$

have the following properties:

(1) intervals are disjoint.
(2) the weightiness of any interval is $\frac{a_{k_\xi} - a_1}{\beta^n}$
(3) their union is included in:

$$\left[\frac{a_1}{\beta-1}, \frac{a_1}{\beta-1} + \frac{(\beta^n - 1)}{\beta^n(\beta-1}(a_m - a_1)\right)$$

Note that $\sum k_\xi = S(n) - S(n-1)$: we have that

$$\frac{1}{\beta^n}(S(n) - S(n-1)) = \frac{1}{\beta^n}\sum k_\xi \leq \frac{1}{\beta^n}\sum a_{k_\xi} - a_i \leq \frac{(\beta^n - 1)}{\beta^n(\beta-1)}(a_m - a_1)$$

thus

$$S(n) - S(n-1) \leq \frac{(a_m - a_1)}{(\beta-1)}(\beta^n - 1)$$

Since S(0) = 1,

$$S(n) - S(0) = \sum_{1=1}^{n} S(i) - S(i-1) \leq \frac{(a_m - a_1)}{(\beta-1)}\sum \beta^i = \left(\frac{(a_m - a_1)}{(\beta-1)}\right)\frac{\beta^{n+1} - \beta}{\beta - 1}$$

thus

$$S(n) \leq \left(\frac{a_m - a_1}{\beta - 1}\right)\frac{\beta^{n+1}}{\beta - 1}$$

Now, consider the sequence of $S(n)$ reals: $\frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_n}{\beta^n}$. We have that the distance between two consecutive terms does not exceed $\frac{a_m - a_1}{\beta^n(\beta-1)}$ and the sequence is distribuited on the interval $\left[\frac{a_1}{\beta-1}, \frac{a_1}{\beta-1} + \frac{(\beta^n-1)}{\beta^n(\beta-1}(a_m - a_1)\right)$, thus

$$(S(n) - 1)\frac{a_m - a_1}{\beta^n(\beta-1)} \geq \frac{a_m - a_1}{\beta^n(\beta-1)}(\beta^n - 1)$$

and, equivalently,

$$S(n) \geq \beta^n$$

Now, let $E \subseteq I_\beta$ a measurable set. $T^{-n}(E)$ is composed by $S(n)$ sets, one for each canonical sequence of order n, and each of them has measure lower than $\frac{\mu(E)}{\beta^n}$, in fact:

$$T^{-n}(E) = \cup_\xi \{x = \bar{x}_{\xi,n} + r(x) \in I_\beta \mid \beta^n r(x) \in E\}$$

where $\bar{x}_{\xi,n} = \sum_{k=1}^{n} \frac{\epsilon_i}{\beta^i}$, $(\epsilon_i) = \xi$. Thus

$$\mu(T^{-n}(E)) \leq \frac{S(n)}{\beta^n}\mu(E) \leq \left(\frac{(a_m - a_1)}{(\beta - 1)}\right) \frac{\beta}{\beta - 1}\mu(E) \tag{6}$$

If $\epsilon_n \neq a_1$ we have that measure of $S(n) - S(n-1)$ sets is exactly equal to $\frac{\mu(E)}{\beta^n}$ because remainders cover completely $\frac{1}{\beta^n}E$:

$$\mu(T^{-n}(E)) \geq (S(n) - S(n-1))\frac{1}{\beta^n}\mu(E) \tag{7}$$

Thus from inequalities (6) and (7) we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T^{-n}(E)) \leq \left(\frac{(a_m-a_1)}{(\beta-1)}\right) \frac{\beta}{\beta-1}\mu(E) \tag{8}$$

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} T^{-n}(E)) &\geq \tfrac{1}{n}\left(1 + \textstyle\sum_{k=1}^{n-1} \frac{S(k)-S(k-1)}{\beta^k}\mu(E)\right) \\
&= \tfrac{1}{n}\left(1 + \textstyle\sum_{k=1}^{n-1} S(k)(\frac{1}{\beta^k} - \frac{1}{\beta^{k-1}}) - \frac{S(0)}{\beta}\right)\mu(E) \\
&= \tfrac{1}{n}\left(1 + \textstyle\sum_{k=1}^{n-1} \beta^k(\frac{1}{\beta^k} - \frac{1}{\beta^{k+1}}) - \frac{S(0)}{\beta}\right)\mu(E) \\
&\geq \mu(E)\left(\frac{(\beta-1)}{a_m-a_1}\right)\frac{\beta-1}{\beta} && (9)
\end{aligned}$$

According the theorem of Dunford and Miller (see [**DM46**]) it follows from the upper inequality that for any L-integrable function $g(x)$ the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(T^k x) = g^*(x)$$

exists for almost all x. By Ergodic Theorem, in order to prove that $g^*(x)$ is a constant almost everywhere, we need to prove that T is an ergodic transformation, i.e. such that if E is a measurable set and

a) $\mu(E) > 0$;
b) $T^{-1}E = E$;

then $\mu(E) = \mu(I_\beta)$. Suppose that E satisfies conditions 3 and 3 and let us consider the class of intervals

$$I_\xi^n = \left[\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i}\right),$$

where $\{\epsilon\}_i$ and $\{\epsilon^+\}_i$ are greedy expansions. By representation Theorem 1 we have that any subinterval of I_β is finite or enumerable union of I_ξ^n .

Now

$$\sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in E \text{ if and only if exists } \{\epsilon_i^x\}_{i=1}^n \text{ and } n \geq 1 \text{ such that } \sum_{i=1}^{n} \frac{\epsilon_i^x}{\beta^i} + \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^{n+i}} \in E,$$

and

$$E = \cup I_{\xi_k}^n \text{ for some } n \text{ and for some sequence } \xi_k,$$

thus

$$E = \left(\sum_{i=1}^{n_a} \frac{\epsilon_i^a}{\beta^i} + \sum_{i=1}^{\infty} \frac{a_1}{\beta^{n_a+i}}, \sum_{i=1}^{n_b} \frac{\epsilon_i^b}{\beta^i} + \sum_{i=1}^{\infty} \frac{a_1}{\beta^{n_b+i}} \right) \sum_{i=1}^{n_b-1} \frac{\epsilon_i^b}{\beta^i} + \frac{\epsilon_{n_b}^b -}{\beta^{n_b}} + \sum_{i=n_b+1}^{\infty} \frac{a_1}{\beta^i} \in E;$$

$$\Rightarrow \sum_{i=n_b+1}^{\infty} \frac{a_1}{\beta^i} \in T^{-n_b} E \Rightarrow \sum_{i=n_b+1}^{\infty} \frac{a_1}{\beta^i} \in E$$

thus for any greedy expansion $\{\epsilon\}_i$ and $\forall n \geq 1$

$$\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^{n+i}} \in E$$

so we can say that $\mu(E) = \mu(I_\beta)$, and the ergodicity of T is proved: $g^*(x)$ is a constant for almost all x.

In order to prove the second part of the theorem let us consider

$$\nu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^k E) = \int_{\frac{a_1}{\beta-1}}^{\frac{a_m}{\beta-1}} \frac{1}{n} \sum_{k=0}^{n-1} E(T^k x)) dx,$$

where $E(x)$ is the characteristic function of E. $E(x) \leq 1$ and Lebesgue Theorem implies that putting $g(x) := E(x)$ the limit $\nu_n(E) \to \nu(E)$ exists. Let us see that ν_n and ν are measures: both ν_n and ν are clearly not negative. We have to prove that they are sub-additive; let E_i a collection of subsets of I_β μ measurable:

$$\begin{aligned} \nu_n(\cup E_i) &= \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^k(\cup E_i)) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \mu(\cup(T^k E_i)) \\ &\leq \sum_i \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^k E_i) \\ &= \sum_i \nu_n(E_i). \end{aligned}$$

By the permanence of sign we have that ν is a measure too. Moreover it is T-invariant:

$$\nu_n(T^{-1}E) = \frac{n+1}{n} \nu_{n+1}(E) - \frac{1}{n} \mu(E) \text{ and for } n \to \infty$$

$$\nu(T^{-1}E) = \nu(E).$$

The equivalence with μ follows from inequalities 8 and 9 and from permanence of sign:

$$\left(\frac{\beta-1}{a_m - a_1} \right) \frac{\beta-1}{\beta} \mu(E) \leq \nu(E) \leq \left(\frac{(a_m - a_1)}{(\beta-1)} \right) \frac{\beta}{\beta-1} \mu(E)$$

and this concludes the proof. □

DEFINITION 8 (Renyi's measure). *Measure ν in previous Theorem is called* Renyi's measure

We concentrate on the problem of existence of a continuum of expansions for almost all $x \in I_\beta$, $1 < \beta < 2$.

The idea is the same as in Sidorov's original theorem ([**Sid03**]): using the existence of an μ-equivalent and T-invariant measure and the ergodicity of T, we apply Poincaré Recurrence Theorem to prove that almost all $x \in I_\beta$ as an occurrence of a string in its greedy expansion which can be substituted with another one in order to obtain two distinct expansions of the same x. Applying Poincaré Recurrence Theorem in strong form, we have that the occurrences of such strings are enumerable in the greedy expansion of almost all $x \in I_\beta$. Now, we can decide to substitute a string or not: this arbitrariety gives $2^{\mathbb{N}}$ different expansions for almost all $x \in I_\beta$.

DEFINITION 9 (cylinders). *Let* $(c_1, \cdots, c_n) \in A^n$ *: we denote as* $[c_1, \cdots, c_n]$ *the* cylinder *of* $(c_1, \cdots, c_m)$ *, i.e. the set of the sequences* (ϵ_i) *such that* $\epsilon_1 = c_1, \cdots, \epsilon_n = c_m$*.*

LEMMA 4. *Let* $x \in I_\beta$ *and assume that its greedy expansion is in the form*

$$(\epsilon_1, \cdots, \epsilon_n, a_m, \underbrace{0, \cdots, 0,}_{m-1 \text{ times}} \epsilon_{n+m+1}, \cdots)$$

where $m = m(\beta) \geq 2$ *is such that*

$$1 + \frac{1}{(\beta-1)\beta^{m-1}} < \frac{1}{\beta-1} \text{where } m \geq \left[\log_\beta \frac{1}{2-\beta}\right] + 1 \geq 2. \tag{10}$$

Then x has at least two distinct expansions.

LEMMA 5. *For any* $1 < \beta < 2$ *, almost every* $x \in I_\beta$ *has denumerable different expansions.*

PROOF. Let us show that for a.e. $x \in I_\beta$ some tail of its greedy expansion belongs to the cylinder $\left[a_m 0^{m-1}\right]$.

Let

$X_\beta$be the space of sequences in alphabet A that represent a greedy expansion in I_β

$\pi_\beta : \; X_\beta \to I_\beta$

$(\epsilon_1, \epsilon_2, \cdots) \to \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}$

By Theorem 7 we can see that there is a measure $\overline{\nu}_\beta$ on X_β equivalent to the π_β - preimage of Lebesgue measure μ, $\overline{\nu}_\beta$ is equivalent to $\mu \circ \pi_\beta$ since:

1. $\mu \circ \pi_\beta$ is a measure because π_β is one-to-one a.e. in X_β
2. By Theorem 7 we have the existence of the Renyi's measure ν equivalent to μ . So we can define $\overline{\nu}_\beta := \nu_\beta \circ \pi_\beta$
3. ν_β is equivalent to $\mu \circ \pi_\beta$ because $\nu_\beta(B) = 0 \leftrightarrow \nu \circ \pi_\beta(B) = 0 \leftrightarrow \mu \circ \pi_\beta(B) = 0$ from the equivalence between ν_β and μ .

Moreover $\overline{\nu}_\beta$ is preserved by the one-sided shift defined as follows:

$$\tau_\beta(\epsilon_1, \epsilon_2, \epsilon_3, \cdots) = (\epsilon_2, \epsilon_3, \cdots)$$

i.e.

$$\nu_\beta(\tau_\beta(B)) = \nu \circ \pi_\beta(\tau_\beta(B)) = \nu(T(\pi_\beta(B))) = \nu(\pi_\beta(B) = \nu_\beta(B),$$

for any measurable $B \in \mathcal{B}$ because Renyi's measure is T invariant.

Finally, again from Renyi, we have that $\overline{\nu}_\beta$ is positive on evey rcylinder in X_β : recall that a cylinder is the set of sequence which first digits are fixed. Now, if the image of a cylinder by π_β has positive Lebesgue measure, then, by the equivalence of μ with ν , ν_β has positive measure too. But the π_β image of any cylinder $[\epsilon_1, \cdots, \epsilon_n]$ is

$$\left[\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{0}{\beta^i}, \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i}\right]$$

and it has clearly positive measure. In particular $\overline{\nu}_\beta \left[a_m 0^{m-1}\right] > 0$, where m satisfies condition of lemma 5, so we can apply Poincaré recurrence Theorem in strong form to

X_β	where $\overline{\nu}_\beta(X) < \infty$ because it is equivalent to $\mu \circ \pi_\beta$ and $\mu \circ \pi_\beta(X_\beta) = \frac{a_m - 0}{\beta-1}$
τ_β	measure preserving with respect to ν_β
$\pi_\beta([a_m 0^{m-1}])$	with positive measure

and see that a.e. point $\bar{x} \in \pi_\beta([a_m 0^{m-1}])$ enumerable occurrences of the string $[a_m 0^{m-1}]$ in its expansion.

Let E' the set of such x. Clearly $\mu(E') = \mu(\pi_\beta([a_m 0^{m-1}])$. Now let

$$E = E' \cup \{x \in I_\beta \mid \exists n \geq 1 such that T^n x \in \pi_\beta([a_m 0^{m-1}])\}$$

we have that

$$\mu(E) = \mu\left(\cup_{n\geq 0} T^{n-1}([a_m 0^{m-1})\right)$$

Moreover $T^{-1}(E) = T(E)$ and as shown in Theorem 7 T is ergodic, thus

$$\begin{aligned}\mu(I_\beta) &= \mu(E)\\ &= \mu\left(\cup_{n\geq 0} T^{n-1}([a_m 0^{m-1})\right)\end{aligned}$$

Applying again Poincaré Theorem to $\cup_{n\geq 0} T^{n-1}([a_m 0^{m-1}])$ we have that the greedy expansion of a.e. x in I_β has enumarable occurrences of the string $\underbrace{(a_m, 0, \ldots, 0}_{m-1 \text{ times}}$, thus it has enumerable different expansions. □

COROLLARY 1. *By Lemma 5 and Theorem 8 we have that any fixed and finite sequence occurs infinite times in any expansion of a.e. real number in I_β.*

THEOREM 8. *For any $1 < \beta < 2$, a.e. $x \in I_\beta$ has a continuum of different expansions.*

PROOF. Let us define

$$\delta^n(\{\epsilon_i\}) = \{\epsilon'_i\}$$

where

$$\{\epsilon'_i\} = \begin{cases} \epsilon'_i = \epsilon_i & i = 1, \cdots, j_n - 1 \\ \epsilon'_{j_n} = 0 \\ \epsilon'_i = \bar{\epsilon}_i & i > j_n \end{cases}$$

and j_n corresponds to the n-th occurrence of the string $\underbrace{(a_m, 0, \cdots, 0)}_{m-1 \text{ times}}$.

Let us observe that $\epsilon'_{i i > j_n}$ is greedy, thus we can apply $\delta^{j_{n_0}+n_1}(\delta^{n_0}(\{\epsilon_i\}\}))$ for $n_1 > 0$ Iterating we get a class of expansions for almost every $x \in I_\beta$ given by:

$$\begin{cases} f_0 = \delta^{n_0}(\{\epsilon_i\} \\ f_k = \delta^{j_{n_k}+n_{k+1}}(f_{k-1})\{\epsilon_i\} \end{cases}$$

Since n_j is arbitrary we obtain thesis. □

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