Topological degree theory and VMO maps

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Let X and Y be two oriented manifolds, let $f : X \to Y$ be a continuous function and let y be a given point in Y; we can associate to f an integer, which we call $d(f, X, y)$, such that:
d1) $d(id, X, y) = 1 \ \forall y \in X$.
d2)

$$d(f, X, y) = d(f, X_1, y) + d(f, X_2, y)$$

whenever X_1 and X_2 are disjoint open subsets of X, such that $y \notin f(X \setminus (X_1 \cup X_2))$.
d3) $d(h(t, .), X, y(t))$ is indipendent of $t \in J = [0, 1]$ whenever $h : J \times \overline{X} \to Y$ and $y(t) : J \to Y$ are continuous and $y(t) \notin h(t, \partial X)$ for all $t \in J$; in other words, the degree is homotopy invariant.
d4) If $d(f, X, y) \neq 0 \Rightarrow f^{-1}(y) \neq \emptyset$.
d5) $d(f, X, y)$ is continuous in f and y. In particular, if X is without boundary and Y is connected, it does not depend on $y \in Y$; hence we can write $d(f, X, y) = d(f, X, Y)$.
• These properties make of the degree a useful tool to study equations of the form $f(x) = y$. Tipically, analysts use degree in the following way. If they want to prove that the equation $f(x) = y$ has solutions, then they look for a " simple " map g homotopic to f; if they can prove that $d(g, X, y) \neq 0$, then $f(x) = y$ has at least a solution by d3) and d4).
• In 1995 H. Brezis and L. Nirenberg have shown that the continuity property 5) is still true in the BMO topology; namely if f and g are continuous maps and $dist_{BMO}(f, g)$ is sufficiently small, then $d(f, X, y) = d(g, X, y)$. Using this fact, they extended the definition of degree to the space VMO, the completion of continuous maps in the BMO-norm.
• Now we recall the definition of the space BMO, Bounded Mean Oscillation: Let

Q_0 be a cube in $\mathbb{R}^n$, and let $f \in L^1(Q_0)$; we set

$$⨍_Q f(x)dx = \frac{1}{|Q|}\int_Q f(x)dx \ \ and \ \ \overline{f}_Q = ⨍_Q f(x)dx.$$

We say that $f \in BMO(Q_0, \mathbb{R})$ if for every cube $Q \subseteq Q_0$ we have that

$$\|f\|_{BMO(Q_0,\mathbb{R})} = \sup_{Q \subset Q_0} ⨍_Q |f(x) - \overline{f}_Q| dx < \infty. \quad (1)$$

It is easy to see that $\|f\|_{BMO(Q_0,\mathbb{R})}$ is a seminorm; we define the BMO-norm of f by

$$\|f\|_{\sim} = \|f\|_{BMO(Q_0,\mathbb{R})} + \|f\|_{L^1}.$$

The space BMO is complete under this norm.
Let us look at some properties of BMO.
• Directly by the definition, we get that $L^\infty \subset BMO$; on the other hand in section 2.3 we shall prove the John-Nirenberg inequality which implies that, if $f \in BMO(Q_0, \mathbb{R})$ and $t > 0$ then

$$|\{x \in Q_0 : |f(x)| > t\}| \leq c_1 e^{\frac{-c_2}{\|f\|_{BMO}}} |Q_0|$$

for some constants c_1 and c_2. As a consequence we have that

$$BMO \subset L^p \ \ \forall p \geq 1.$$

In particular,

$$L^\infty \hookrightarrow BMO \hookrightarrow L^p \ \ \forall p \geq 1.$$

• The space BMO is strictly larger than L^∞, indeed we shall see in section 2.3 that the function $\log|x| \in BMO \setminus L^\infty$.
• Concerning VMO, its name comes from Vanishing Mean Oscillation; indeed, a theorem of Sarason (section 2.4) implies that $f \in VMO$ iff a stronger property than (1) holds, i.e iff, denoting by $Q(x, \epsilon)$ the cube centered in x with side ϵ we have

$$\lim_{\epsilon \to 0} ⨍_{Q(x,\epsilon)} |f(x) - \overline{f}_{Q(x,\epsilon)}| dx \to 0 \ \ uniformly \ in \ x.$$

It is easy to see that $L^\infty(Q_0,\mathbb{R}) \not\subseteq VMO(Q_0,\mathbb{R})$, indeed, an easy calculation showes that the function $\mathbb{1}_{(0,1)}$ does not satisfy the formula above for $Q_0 = (-1,1)$. But of course $C^0(\Omega,\mathbb{R}) \subset VMO(\Omega,\mathbb{R})$. In Section 2.6 we show that the inclusion is strict. The example is the function $f(x) = \log|\log|x||$ which belongs to $VMO(\Omega,\mathbb{R})$ for any Ω a bounded domain in $\mathbb{R}^n$ containing the origin. On the other side, VMO is strictly containeed in BMO, because we shall see in section 2.6 that $f(x) = \log|x| \notin VMO$.

• Let Ω be a bounded open domain in $\mathbb{R}^n$; some important spaces which embed in $VMO(\Omega)$ are :

$$W^{1,n}(\Omega) = \{f \in L^n(\Omega) : \nabla f \in L^n(\Omega)\}$$

and

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy < \infty \right\} \tag{2}$$

for all $0 < s < 1$, $1 < p < \infty$ with $sp = n$, (see Section 2.6).

Definition 2 is Gagliardo's characterization of the Sobolev space with fractional exponent.

• Naturally the definition of BMO and VMO can be extended to a function in $L^1(Q_0,\mathbb{R}^n)$. In other words $f \in BMO(Q_0,\mathbb{R}^n)$ ($VMO(Q_0,\mathbb{R}^n)$) if each component of f is $BMO(Q_0,\mathbb{R})$ ($VMO(Q_0,\mathbb{R})$).

• Let X be a smooth n-dimensional connected compact Riemannian manifold without boundary, we shall see in section 2.1 that we can define $BMO(X,\mathbb{R}^n)$ and $VMO(X,\mathbb{R}^n)$.

• Finally let Y be a compact manifold without boundary; we shall always suppose that Y is smoothly embedded in some $\mathbb{R}^n$. We say that f belongs to $BMO(X,Y)$ ($VMO(X,Y)$), if $f \in BMO(X,\mathbb{R}^n)$ ($VMO(X,\mathbb{R}^n)$) and $f(x) \in Y$ a.e.

• We shall see in section 2.2 that a different choice of the Riemannian metric on X or of the embedding of Y yields an equivalent metric.

• Given a manifold Y embedded in $\mathbb{R}^n$, in a neighbourhood of Y we can define a projection operator, which associates to $y \in \mathbb{R}^n$ the unique point on Y closest to y. If $f \in VMO(X,Y)$, for $\epsilon > 0$ small, we can define

$$\overline{f}_\epsilon(x) := \fint_{B_\epsilon(x)} f(y)d\sigma(y) \ \ and \ \ f_\epsilon(x) := P \circ \overline{f}_\epsilon(x).$$

Since f_ϵ is a continuous map, $d(f_\epsilon, X, y)$ is well defined $\forall y \in Y$. If X, Y are manifolds without boundary and Y is connected, we have by d5) that $d(f_\epsilon, X, y)$ does not depend on $y \in Y$, hence $d(f_\epsilon, X, y) = d(f_\epsilon, X, Y)$. We note that, if ϵ is small $d(f_\epsilon, X, Y)$ does not depend on $\epsilon \in (0, \epsilon_0)$. To show this we use the fact that the degree is invariant by homotopy; in particular using the deformation $f_{t\epsilon+(1-t)\epsilon'}$, for ϵ, ϵ' small, $0 < t < 1$, we see that $d(f_\epsilon, X, Y) = d(f_{\epsilon'}, X, Y)$. Thus we can define

$$d(f, X, Y) := \lim_{\epsilon \to 0} d(f_\epsilon, X, Y). \tag{3}$$

• We shall see in section 4.1 that the definition 3 is indipendent of the choice of Riemannian metric on X and of the embedding of Y.

• Some properties of VMO-degree are :

1) $d(id, X, Y) = 1$.

2)Let $f \in VMO(X, Y)$. Then there exists $\delta > 0$ depending on f, such that if $g \in VMO(X, Y)$ and

$$dist(f, g) < \delta,$$

then

$$d(f, X, Y) = d(g, X, Y).$$

3)Let $H_t(\cdot)$ be a one parameter family of VMO maps X to Y, depending continuously in the BMO topology, on the parameter t. Then $d(H_t(\cdot), X, Y)$ is indipendent of t.

4) If $d(f, X, Y) \neq 0$ then the essential range of f is Y. We define the essential range to be the smallest closed set Σ in Y such that $f(x) \in \Sigma$ $a.e.$

•Recall that if f and $g \in C^0(X, Y)$, there is a uniform $\delta > 0$ such that

$$|f - g|_{C^0} < \delta \Rightarrow d(f, X, Y) = d(g, X, Y).$$

Surprisingly if f and $g \in VMO(X, Y)$, δ depends on f. In Chapter 4 we give an example building two maps f, g from S^1 to S^1 arbitrarily close in the $H^{\frac{1}{2}}(S^1)$ topology, and thus in BMO topology, but with different degrees.

• One can ask whether it is possible to define the degree for maps in $L^p(X, Y)$, $1 \leq p \leq +\infty$, and $BMO(X, Y)$. The answer is negative; indeed in Section 4.1 we prove that these spaces are arcwise connected.

• In section 1.8 we recall that if $f \in C^1(X, Y)$, then

$$d(f, X, Y) = \frac{1}{|Y|} \int_X \det J_f(x) d\sigma(x) \tag{4}$$

where $|Y|$ denotes the volume of Y.

But we have seen that $W^{1,n}(X, Y) \hookrightarrow VMO(X, Y)$, and the integral above, converges for $f \in W^{1,n}$; we show in section 4.2 that (4) holds when $f \in W^{1,n}$ and d is the VMO degree.

• If $X = Y = S^n$, where S^n is the sphere of $\mathbb{R}^n$, a famous theorem of Hopf says that two maps in $C^0(S^n, S^n)$ are homotopic iff they have the same degree. In section 4.1 we show that this holds also for VMO maps.

• An interesting case is when we consider $f \in H^{\frac{1}{2}}(S^1) = W^{\frac{1}{2},2}(S^1, S^1)$. From Gagliardo's characterization we have

$$H^{\frac{1}{2}}(S^1) = \left\{ f \in L^2(S^1) : \int_{S^1} \int_{S^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} dxdy < +\infty \right\}.$$

We also recall the characterization of $H^{\frac{1}{2}}(S^1)$ in terms of the Fourier coefficents $(\hat{f}_n)$ of f :

$$H^{\frac{1}{2}}(S^1) = \left\{ f \in L^2(S^1) : \sum_{n=-\infty}^{+\infty} (1 + |n|)|\hat{f}_n|^2 < +\infty \right\}. \tag{5}$$

Finally we recall that if $f \in C^1(S^1, S^1)$, by a well-know formula of complex analysis we have that

$$d(f, S^1, S^1) = \frac{1}{2\pi i} \int_f \frac{dz}{z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(\theta)}{f(\theta)} d\theta = \frac{1}{2\pi i} \int_0^{2\pi} f'(\theta)\overline{f}(\theta) d\theta \tag{6}$$

where we have used the fact that $\frac{1}{f(\theta)} = \overline{f}(\theta)$ because $\|f(\theta)\| = 1$.

Now we consider the Fourier expansion of f :

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \hat{f}_n e^{in\theta}.$$

Inserting it into 6, we find

$$d(f,S^1,S^1) = \frac{1}{2\pi i}\int_{S^1}\left[\sum_{n=-\infty}^{+\infty}\overline{\hat{f}_n}e^{-in\theta}\sum_{m=-\infty}^{+\infty}(im)\hat{f}_m e^{im\theta}\right]d\theta = \sum_{n=-\infty}^{+\infty} n|\hat{f}_n|^2. \quad (7)$$

The density of $C^1(S^1,S^1)$ into $H^{\frac{1}{2}}(S^1,S^1)$ and the continuity of the degree yield that formula (7) holds also when $f \in H^{\frac{1}{2}}(S^1,S^1)$.

•This fact has a surprising consequence :

Let (a_n) be a sequence of complex numbers satisfying

$$\sum_{n=-\infty}^{+\infty} |n||a_n|^2 < +\infty \quad (8)$$

$$\sum_{n=-\infty}^{+\infty} |a_n|^2 = 1 \quad (9)$$

and

$$\sum_{n=-\infty}^{+\infty} a_n\overline{a}_{n+k} = 0 \ \ \forall k \neq 0 \quad (10)$$

Then

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2 \in \mathbb{Z} \quad (11)$$

• If f is only continuous, the series 7 is not convergent, but the Fourier coefficents still exist.

One problem proposed by Brezis is whether one can hear the degree of continuous maps. In other words, if f and g are continuous maps and

$$|\hat{f}_n| = |\hat{g}_n| \ \forall n \in \mathbb{Z}$$

can one conclude that $d(f,S^1,S^1) = d(g,S^1,S^1)$?

J. Bourgain and G. Kozma [5] have shown that the answer is negative. They have constructed a complicated example of two continuous maps f and g of the circle to itself with $|\hat{f}_n| = |\hat{g}_n| \ \forall n \in \mathbb{Z}$ but with different degree.

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