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Synthesis

Stochastic Methods for the Evaluation of Investment Plans

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Synthesis

The goal of this work is to give an introductory survey to the theory of *real options*. About four decades ago, around 1977, professor Stewart Myers, at MIT Sloan School of Management, coined the term *real options*, observing that corporate investment opportunities can be viewed as *call options* on real assets. Otherwise saying, a company having an opportunity to invest is holding much like a financial call option: it has the right but not the obligation to buy an asset, namely, the entitlement to the stream of profits from the project, at a future time of its choice. When a company makes irreversible investment expenditure, it actually *exercises* this call option. Hence, the real options approach applies financial options theory to investment decisions under uncertainty. Since then, numerous researchers addressed investment opportunity under uncertainty by using a real options approach, which has become increasingly popular. A large body of research has established a theoretical framework for modeling and pricing real options and a variety of real options have been investigated. Since the early stages of their introduction, the applications of real options have been extended from natural resources investment to a wide range of investment problems. The business community have also shown a growing interest in real options. Indeed, many world famous companies have adopted the techniques of real options for project valuation and investment decision-making.

Real options theory can be described as a new valuation, project management, and strategic decision making paradigm that replaces many of the traditional methods, by allowing for flexible or staged decisions under uncertainty (Trigeorgis, [28] 1996). This is also where the real options theory deviates from the traditional discounted cash flow method (DCF). Actually, DCF method assumes that the company has to accept all the possible outcomes of a project as a whole, once the investment has been decided. Furthermore, DCF method views any investment as a *now-or-never* opportunity, while in real option theory, the investor may wait for some time until additional favorable information validates the investment commitment. Real options are embedded in many assets and projects, although the value of these options is not always recognized. The risk of underestimating the asset value always exists. In traditional investment theory under certainty, there is no option value and investment is made just following the simple Net Present Value rule: invest when the present discounted value of the investment equals or exceeds the investment cost. The basic formula for computing the NPV of a project is given by

$$NPV = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t} - I_0,$$

where I_0 is the initial investment outlay, r is the *rate of return*, yielding the *discount rate*, and c_t is the future cash inflow at time t.
A central question of capital budgeting concerns the specification of an appropriate *rate of return* or the corresponding *discount rate*. The first, known as *required rate of return*, represents the time value of money and the relative risk of the project in the discounted cash flow model. If the cash flows generated by the project under consideration were known for certain, the required rate of return would be the *risk free interest rate*. However, the future cash flows for projects are usually uncertain. The uncertainty is then incorporated into the analysis by using a *risk adjusted rate of return*. Hence, the formula for computing the NPV has to be modified as follows

$$NPV = \sum_{t=1}^{T} \frac{\mathbb{E}[c_t]}{(1+r_a)^t} - I_0,$$

where $r_a > r$ is the risk-adjusted discount rate, which accounts for the uncertainty in the sequence of the future cash inflows, and $\mathbb{E}[\cdot]$ is the ordinary expectation operator. Note that r_a is defined as the sum of the risk-less interest rate r, which is used to discount for the time value of money (pure discount) and a discount *risk premium* ϕ, that is

$$r_a = r + \phi.$$

According to the common knowledge, the most relevant mislead in the basic NPV method is the implicit lack of flexibility in management' choices. Otherwise saying, the method assumes management's passive commitment to a certain *operating strategy*, which is usually not the case. The basic NPV method also ignores the synergy effects that the investment project can create. A project of a certain kind might allow the company to expand into a second project, which would not have been possible without the first project (e.g., many research and development projects). It is the value of this second project that NPV ignores. The *decision tree analysis* (DTA) carries NPV method a little further. Instead of presuming a single scenario of future cash

flows, many different scenarios are considered. By solving the problem in this way, several possibilities of futures states of the world and also the set of decisions made each time in each state will be incorporated into the analysis. The future cash flows and probabilities used in the analysis reflect the information available to the company at the present time. The values are derived from the basis of past information (Brigham & Gapenski [4] 1996). Using DTA in combination with NPV is an attempt to tackle the uncertainty of future cash flows by making different scenarios to be dealt with the NPV approach. In fact, DTA incorporates into the analysis the issues concerning flexibility mentioned above. This makes DTA analysis a better tool than basic NPV to evaluate projects. However, to find the appropriate required rate of return is a problem in both basic NPV and DTA. A seemingly small difference in the discount rate can have a huge impact on the overall result. Actually, Trigeorgis [28] (1996) argues that the most serious problem in DTA analysis is to find the appropriate discount rate. This is because the presence of flexibility would alter the project's risk, hence altering the discount rate that would prevail without the flexibility. For example the possibility to abandon the project would clearly reduce the project's risk and lower the discount rate. Then, using the same discount rate as in a basic NPV would lead to undervalue the project. Cortazar [10] (1999) supports this argument by emphasizing that whichever pricing model is used (CAPM or APT) most investment projects will find their risk structure change over time. This means that the risk-adjusted discount rate also will change over time, which in turn will lead to errors in the result. Another critique of the DTA analysis refers to its complexity in the sense that when it is applied in most realistic investment settings, it will easily turn out to be a unmanageable *decision-bush analysis*, as the number of paths through the tree expands geometrically with the number of decisions, outcome variables, or states considered for each variable (Trigeorgis, [28] 1996).
In the option value theory of investment, the fact that investments are irreversible and undertaken under uncertainty leads the firm to consider an additional component in its investment choice: the value of waiting to invest, with the aim of reducing the uncertainty on the future. A growing body of research shows that the ability to delay irreversible investment expenditures can profoundly affect the decision to invest. For analyzing investment decisions, it is needed to establish a rich framework that enables managers to address the issues of *irreversibility*, *uncertainty*, and *timing* directly. In fact, one of the most important aspects of investments decision making is the timing of the investment and the flexibility involved. Not only is the investment opportunity itself important, but also managers' ability to decide how to exploit those opportunities most effectively to increase shareholders'

value. The managerial flexibility inherent in real investment decisions is valuable when the economic environment is uncertain and investment decisions are irreversible. Flexibility is the ability to defer, abandon, expand, or contract an investment. The theory of real options is based on an important analogy with financial options. So the problem of how to exploit an investment opportunity reduces to this: how can a company exercise an option optimally? Scholars and financial professionals have been studying the valuation and optimal exercising of financial options for the past two decades. Therefore, the theory of financial options is the cornerstone for the theory of real options.

In our work, the goal is more specifically to describe the essential mathematical tools that are used in real option theory: *stochastic calculus*, *dynamic programming*, and *contingent claims analysis* and to present typical models of the real option approach to investment evaluation under uncertainty. Stochastic processes combine dynamics with uncertainty. In a dynamic model without uncertainty, the current state of the system will determine its future state. When uncertainty is add, the current state determines only the probability distribution of future state, not the actual value.

The *Brownian motion with drift* is the basic stochastic process in Economics and Finance. The stochastic equation for its variation in time is

$$dX_t = \alpha\, dt + \sigma\, dW_t, \tag{1}$$

where dW_t is the differential of the Wiener process, which can be characterized as

$$dW_t = \epsilon_t \sqrt{dt} \qquad \epsilon_t \approx N(0,1)$$

In Equation (1), α is called the *drift parameter*, and σ the *volatility parameter*. Note that over any infinitesimal time interval dt, the change dX_t is normally distributed, with expectation $\mathbb{E}(dX_t) = \alpha dt$ and variance $\mathbb{V}ar(dX_t) = \sigma^2 dt$. As an important consequence of the definition, a Brownian motion turns out to be a *Markov process*, which implies that only the current information is useful in forecasting the future path of the process. Nevertheless, the most exploited processes are likely the *geometric Brownian motion* and the *mean revererting process*. The geometric Brownian motion with drift is given by

$$dX_t = \alpha X_t\, dt + \sigma X_t\, dW_t. \tag{2}$$

where X_0 is know, the percentage changes dX_t/X_t are normally distributed. Since these are changes in the natural logarithm of X_t, the absolute changes dX_t are *lognormally* distributed. We will show that, if X_t is given by Equation (2), then $F(t, X_t) = log(X_t)$ is the following simple Brownian motion with drift obtained via Itô's Lemma

$$dF(t, X_t) = \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t. \tag{3}$$

Hence, over a finite time interval t, the change in the logarithm of X_t is normally distributed with mean $\left(\alpha - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$.
Dixit and Pindyck [12] argue that uncertainty on the value of a new technology can be modeled as a *geometric Brownian motion.*
Brownian motions tend to wander far from their starting points. This is not realistic for some economic variables, for example, the prices of raw commodities such as copper or oil. Despite such prices are often modeled as geometric Brownian motions, one could argue that they should somehow be related to long-run marginal production costs. In other words, while in the short run the price of oil might fluctuate randomly up and down (in response to wars or revolutions in oil producing countries, or in response to the strengthening or weakening of the OPEC cartel), in the longer run it ought to be drawn back towards the marginal cost of producing oil. Thus, one might more realistically argue that the price of oil should be modeled as a *mean-reverting process.* The simplest mean-reverting process, also known as an *Ornstein-Uhlenbeck process*, satisfies the stochastic differential equation

$$dX_t = \eta(\bar{X} - X_t)\, dt + \sigma\, dW_t, \tag{4}$$

where $\bar{X}$ is the *normal* level of X_t, that is the level to which X_t tends to revert and η is the *speed of reversion.* Note that the expected change in X_t depends on the difference between X_t and $\bar{X}$. If X_t is higher [resp. lower] than $\bar{X}$, it is more likely to fall [resp. rise] over the next spot interval of time. Hence, this process, although satisfying the Markov property, does not have independent increments.
Based on the mathematical tools sketched above, the milestone of for both financial option valuation is the so called *Black & Scholes Formula* for the pricing of a call option on an underlying asset whose dynamics follows a geometric Brownian motion. A call option is a contract between two parties, a *holder* and a *writer.* By paying a *prime* to the writer, the holder acquires the right, but not the obligation, to buy from the writer one unit of an underlying asset with price X_t, within a predetermined date T, called *maturity* or *expiration date*, at a predetermined price K, called *strike-price.* The writer, upon the payment of the prime, takes the obligation to satisfy the holder's right upon the exercise of the option. The Black & Scholes pricing formula relies on the following assumptions: the short-term interest rate is constant; the underlying asset price follows a geometric Brownian motion

and pays no dividend; the option is European, that is can be exercised only at maturity. Moreover, to allow a continuous trading, the market is assumed to be frictionless: there are no transactions costs or taxes, no restrictions on short sales, such as margin requirements, all shares of all securities are infinitely divisible, and borrowing and lending are unrestricted. The additional assumption of the completeness of the market, that is the possibility of hedging all derivatives via a replicating portfolio composed by the risky asset underlying the derivative and a risk free asset paying the short-term interest rate, existence of a risk-neutral probability distribution, allows to write the expected value $C(t,x)$ of a European call option given the realization x of the underlying X_t at time t, as a solution to the partial differential equation

$$\frac{\partial C(t,x)}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C(t,x)}{\partial x^2} + r\left(x\frac{\partial C(t,x)}{\partial x} - C(t,x)\right), \tag{5}$$

which is the celebrate Black & Scholes equation for the pricing of an European call option. Solving Equation (5) we obtain the Black-Scholes pricing formula

$$C(t,x) = e^{-r(T-t)}\left[xN(d_1)e^{r(T-t)} - KN(d_2)\right],$$

where

$$d_1 \equiv \frac{\log\left[x/K + (r+\frac{1}{2}\sigma^2)T\right]}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log\left[x/K + (r-\frac{1}{2}\sigma^2)T\right]}{\sigma\sqrt{T}}.$$

Here $N(.)$ is the cumulative distribution function of a normally distributed random variable with null mean and unitary standard deviation. As we will show in the sequel, the Black & Scholes formula reveals to be also the main tool for the pricing of the simplest, but most basic, real option that is the option of waiting to invest.

This work is divided into three main parts: Part I contains the real option theory compared with more traditional investment evaluation methods; in Part II we describe general models for managing uncertainty and we show how to tackle the problem of optimal decision under uncertainty. Finally in Part III we present some example illustrating the application of the techniques developed throughout the work.

The approach used is in large part descriptive, in the sense that, it draws on extensive existing knowledge about the problem, which it is well structured in the theory.

Hence, in Part I, the main purpose is to introduce and apply a real option based valuation framework on an investment project that is able to incorporate the value of flexibility in the capital budgeting process, in contrast to traditional capital budgeting analysis based on standard discounted cash flow methodologies. Further, we aim to show how to bridge the gap between the practical problems of application of the real option theory on investment projects, and the sophisticated mathematics associated with financial option pricing theory. Summarizing, the part I is based on a literature review and focuses on traditional capital valuation methods, financial option theory and the fundamentals of real options theory. In particular, Chapter 1 explains the most popular traditional evaluation methods, by means of simple examples. Chapter 2 presents an introduction to financial option theory, which provides the direct link to real option theory. In Chapter 3, we move from financial option theory to real option theory.

In Part II, we give a brief account of the mathematical tools that are exploited in real option theory. In Chapter 4, we begin with simple *discrete-time processes* and then we turn to the *Wiener process*, or *standard Brownian motion*, a continuous-time process which is a fundamental building block for many of the models. We will explain the meaning and properties of the Wiener process and show how it can be derived as the continuous limit of a discrete-time random walk. Then, we will see how the Wiener process can be generalized to a broad class of continuous-time stochastic processes, so called *Itô processes.* Itô processes can be used to represent the dynamics of the value of a project, output prices, input costs, and other variables that evolve stochastically over time and that affect the decision to invest. To deal with these processes, we have to make use of *Itô's calculus.* In this setting, the *Fundamental Theorem of Stochastic Calculus* is an important result that allow us to integrate and, in an appropriate sense, differentiate functions of stochastic processes. We provide a heuristic derivation of *Itô's Lemma* and then show how it can be used to perform simple operations on functions of Wiener processes. Chapter 5 concerns optimal sequential decision under uncertainty. We begin with some basic ideas of the general technique for such optimization: *dynamic programming.* We introduce this in a simple two-period example, and show how the basic ideas extend to more general multiperiod choice problems, where the uncertainty is modeled via the stochastic processes introduced in Chapter 4. We present the fundamental equation of dynamic programming, and indicate methods for solving it with reference to some application. Then, we turn to a market setting, where the risk generated by stochastic process can be hedged by continuous trading of *contigent claims.* We show how the sequential decision can be equivalently handled by constructing a *dynamic hedging strategy*, a portfolio whose composition changes over time to replicate the return and risk characteristic of the real investment.

In Part III, we turn to some example to illustrate applications and extensions of the presented techniques. We begin Section 6.1 with a problem of interest to oil companies, how to value an undeveloped offshore oil reserve, and how to decide when to invest in development and production. As we will see, an undeveloped reserve is essentially an option: it gives the tenant the right to invest in development of the reserve and then produce oil. By valuing this option we can regularly spend hundreds of millions of dollars for offshore reserves, so it is important to determine how to value and best exploit them. Then, in Section 6.2, we turn to an investment timing problem in the electric utility industry. The Clean Air Act calls for the reductions in overall emissions of sulfur dioxin, but to minimize the cost of these reductions, it gives utilities a choice. They can invest in expensive *scrubbers* to reduce to mandated level, or they can buy tradeable *allowances* that let them pollute. There is considerable uncertainty over the future price of allowances, and an the investment in scrubbers is irreversible. The utility must decide whether to maintain flexibility by relying on allowances or invest in scrubbers. We shows how this problem can be addressed using the options approach.

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