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Sobolev inequalities in the limiting case and exponential integrability

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Summary of the thesis

Introduction

For open domains $\Omega \subset \mathbb{R}^N$ the case $p = N$ is considered a limiting case for the embeddings of Sobolev spaces $W_0^{1,p}(\Omega)$; in fact, when $p < N$ the Sobolev space $W_0^{1,p}(\Omega)$ is embedded just in some $L^q(\Omega)$ with $q > p$, precisely

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \quad \text{for} \quad p^* = \frac{Np}{N-p} > p$$

and thus, through interpolation, the same embedding holds for any $q \in [p, p^*]$; on the other hand, functions in $W_0^{1,p}(\Omega)$ for $p > N$ are bounded and even Hölder continuous:

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right) \quad \text{for} \quad \alpha = 1 - \frac{N}{p} > 0$$

Regarding $W_0^{1,N}(\Omega)$, its functions actually belong to any $L^q(\Omega)$ for $q \in [N, +\infty)$, but they are not bounded; however, the known counterexamples of unbounded functions in $W_0^{1,N}(\Omega)$ have only logarithmic singularities, hence one may suspect that a condition of exponential integrability holds.
In fact, Trudinger [Tru67] found that, for bounded domains, it holds

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha u^{\frac{N}{N-1}}} < +\infty \tag{1}$$

for some $\alpha > 0$; this was later refined by Moser [Mos71], who proved that (1) holds if and only if $\alpha \leq \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the $N-1$-dimensional measure of the unit sphere $\mathbb{S}^{N-1}$; Moser also showed that the supremum in (1) depends linearly on the measure of Ω and that for higher exponents the integral is pointwise defined for any $u \in W_0^{1,N}(\Omega)$ even though, for any fixed $\alpha > 0$, it can be made arbitrarily large between functions with the integral of $|\nabla u|^N$ less then 1.

Sobolev embeddings in the limiting case

Exponential integrability and the asymptotic behavior of the best Sobolev constant, defined as

$$S_p(\Omega) = \inf_{0 \not\equiv u \in W_0^{1,N}(\Omega)} \frac{\int_\Omega |\nabla u|^N}{\left(\int_\Omega |u|^p\right)^{\frac{N}{p}}}$$

are actually closely related, as the following lemmas show:

Lemma 1.

Let $\Omega \subset \mathbb{R}^N$ be an open domain.
If the following exponential integrability condition holds

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \le 1} \int_\Omega \left(e^{\alpha |u|^q} - \sum_{j=0}^{k} \frac{\alpha^j |u|^{jq}}{j!} \right) < +\infty$$

for some $\alpha > 0$, $q > 0$, $k \in \mathbb{N}$, then the best Sobolev constant $S_p(\Omega)$ satisfies

$$\liminf_{p \to +\infty} p^{\frac{N}{q}} S_p(\Omega) \ge (e\alpha q)^{\frac{N}{q}}$$

Lemma 2.

Let $\Omega \subset \mathbb{R}^N$ be an open domain.
If the best Sobolev constant $S_p(\Omega)$ satisfies

$$\liminf_{p \to +\infty} p^{\frac{N}{q}} S_p(\Omega) \ge (e\alpha q)^{\frac{N}{q}}$$

for some $\alpha > 0$, $q > 0$, then the following exponential integrability condition holds

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \le 1} \int_\Omega \left(e^{\beta |u|^q} - \sum_{j=0}^{k} \frac{\beta^j |u|^{jq}}{j!} \right) < +\infty$$

for some $k \in \mathbb{N}$ and for any $\beta \in (0, \alpha)$.

The latter result was used by Trudinger in its original proof of (1), where he derives exponential integrability from estimates on the best Sobolev constants; moreover, Mugelli and Talenti [MT98] gave the following estimate on for hyperbolic discs:

Theorem 3.
Let $\Omega = B_1(0) \subset \mathbb{R}^2$ *be the unit ball and* $g_h = \left(\frac{2}{1-|x|^2}\right)^2 g_e$ *the hyperbolic metric.*
Then, it holds

$$S_p(\Omega, g_h) = \inf_{0 \not\equiv u \in H_0^1(\Omega, g_h)} \frac{\int_\Omega |\nabla_{g_h} u|^2 dV_{g_h}}{\left(\int_\Omega |u|^p dV_{g_h}\right)^{\frac{2}{p}}} \geq \left(\frac{8\pi p \Gamma\left(\frac{p}{p-2}\right)^2}{(p-2)^2 \Gamma\left(\frac{2p}{p-2}\right)} \right)^{1-\frac{2}{p}}$$

Hence, applying lemma 2, one gets exponential integrability for conformal metrics on the unit disc, independently from Trudinger and Moser's results:

Corollary 4.
Let $g_\rho = \rho(x) g_e$ *be a conformal metric on* $\Omega = B_1(0) \subset \mathbb{R}^2$ *that is bounded by the hyperbolic one, namely* $\rho(x) \leq C \left(\frac{2}{1-|x|^2}\right)^2$.
Then, it holds

$$\sup_{u \in H_0^1(\Omega, g_\rho), \int_\Omega |\nabla_{g_\rho} u|^2 dV_{g_\rho} \leq 1} \int_\Omega \left(e^{\alpha u^2} - 1\right) dV_{g_h} < +\infty$$

for all $\alpha < \frac{4\pi}{e}$.

On the other hand, from lemma 1, sharp Moser-Trudinger inequality provides information about the optimal Sobolev constant, giving not only a lower bound but a sharp asymptotic behavior, as proved by Ren and Wei [RW95]:

Theorem 5.
Let $\Omega \subset \mathbb{R}^N$ *be an open domain having finite measure.*
Then,

$$\lim_{p \to \infty} p^{N-1} S_p(\Omega) = \omega_{N-1} \left(\frac{N^2 e}{N-1}\right)^{N-1}$$

This result regarding $S_p(\Omega)$ allows to give asymptotic information even on the functions u_p which attain the supremum in Sobolev inequality, whose existence is ensured by the Rellich-Kondrachov compactness theorem for bounded domains.
Adimurthi and Grossi [AG04] found out the following result on some renormalized sequences:

Theorem 6.
Let $\Omega \subset \mathbb{R}^2$ *a bounded domain.*
Defining $x_p \in \Omega$ *as a point such that* $u_p(x_p) = \|u_p\|_{L^\infty(\Omega)}$,

$$\varepsilon_p = \frac{1}{\sqrt{(p-1)S_p(\Omega)}\|u_p\|_{L^\infty(\Omega)}^{\frac{p-2}{2}}} \qquad \Omega_p = \frac{\Omega - x_p}{\varepsilon_p}$$

and

$$z_p(x) = \frac{p-1}{\|u_p\|_{L^\infty(\Omega)}}(u_p(\varepsilon_p x + x_p) - \|u_p\|_{L^\infty(\Omega)}) \in H_0^1(\Omega_p)$$

then, for any subsequence $p_k \underset{k\to+\infty}{\to} +\infty$, *it holds*

$$z_{p_k}(x) \underset{k\to+\infty}{\to} z(x) = \log \frac{1}{\left(1+\frac{|x|^2}{8}\right)^2} \quad \text{in } C^1_{loc}\left(\mathbb{R}^2\right)$$

Using this result, Adimurthi and Grossi also discovered the exact asymptotic value of the supremum norm of u_p, which was estimated from above and below and conjectured by Ren and Wei [RW96].

Theorem 7.
It holds

$$\lim_{p\to+\infty} \|u_p\|_{L^\infty(\Omega)} = \sqrt{e}$$

Ren and Wei [RW94, RW96] studied the blow-up of u_p, showing that, for $p \to +\infty$, the extremal functions asymptotically vanish except for x_p; moreover, they found out some convergence properties of other renormalized sequences.

Theorem 8.
Let $\Omega \subset \mathbb{R}^2$ *a bounded domain.*
Defining

$$v_p = \frac{u_p}{S_p(\Omega)\int_\Omega u_p^{p-1}} \quad \text{and} \quad f_p = \frac{u_p^{p-1}}{\int_\Omega u_p^{p-1}}$$

and, for a sequence $p_k \underset{x\to+\infty}{\to} +\infty$, *the blow-up set of* v_{p_k} *as*

$$S = \left\{x \in \overline{\Omega} : \exists x_k \underset{k\to+\infty}{\to} x, v_{p_k}(x_k) \underset{k\to+\infty}{\to} +\infty\right\}$$

then, the following facts hold, up to a subsequence:

1. *S contains exactly one point, say $S = \{\widetilde{x}\}$.*

2. $$f_{p_k} \rightharpoonup \delta_{\widetilde{x}}$$

   in the sense of distribution.

3. $$v_{p_k}(x) \underset{k \to +\infty}{\to} G_{\widetilde{x}}(x)$$

   in $C^2_{loc}\left(\overline{\Omega} \backslash \{\widetilde{x}\}\right)$ and weakly in $W^{1,q}(\Omega)$ for all $q \in [1, 2)$, where $G_{\widetilde{x}} = G_{\Omega,\widetilde{x}}$ is the Green function of $-\Delta$ on Ω.

4. *$\widetilde{x}$ is a critical point for the Robin function $R(x) = H_x(x)$, where*

   $$H_{\widetilde{x}}(x) = -G_{\widetilde{x}}(x) - \frac{1}{2\pi} \log |x - \widetilde{x}|$$

   is the regular part of $G_{\widetilde{x}}$.

Extensions of Moser-Trudinger inequality

The second part of the thesis is devoted to extensions of Moser-Trudinger inequality, particularly in the case of unbounded domains; first of all, in this case one has to consider, rather than just $e^{\alpha_N |u|^{\frac{N}{N-1}}}$, the integral of

$$\Phi(u) = e^{\alpha_N |u|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\alpha_N^j |u|^{\frac{jN}{N-1}}}{j!}$$

because powers with an exponent lower than N might not be summable.
The first case considered is the unit ball endowed with a conformal metric; a result from Gianni Mancini and Sandeep [MS10] improves corollary 4:

Theorem 9.
Let $g_\rho = \rho(x) g_e$ be a conformal metric on $\Omega = B_1(0) \subset \mathbb{R}^N$.
Then the following conditions are equivalent:

1. *g_ρ is bounded by the hyperbolic metric, that is*

   $$\exists C > 0 \quad \textit{such that } \rho(x) \leq C \left(\frac{2}{1 - |x|^2} \right)^N \quad \forall x \in \Omega$$

2. *Sharp Moser-Trudinger inequality holds for the metric g, that is*

$$\sup_{u \in W_0^{1,N}(\Omega, g_\rho), \int_\Omega |\nabla_{g_\rho} u|^N dV_{g_\rho} \le 1} \int_\Omega \Phi(u) dV_{g_\rho} < +\infty$$

The validity of sharp Moser-Trudinger inequality for metrics bounded by g_h implies that theorem 5 also holds for the Sobolev constants for these metrics; moreover, in the 2-dimensional case, a stronger result, again from Gi. Mancini and Sandeep [MS10], holds for the metrics corresponding to a conformal diffeomorphism with simply connected domains.

Theorem 10.
Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain.
Then, the following conditions are equivalent:

1. *Sharp Moser-Trudinger inequality holds for Ω, that is*

$$\sup_{u \in H_0^1(\Omega), \int_\Omega |\nabla u|^2 \le 1} \int_\Omega \left(e^{4\pi u^2} - 1\right) < +\infty$$

2. *Subcritical Moser-Trudinger inequality holds for Ω, that is*

$$\sup_{u \in H_0^1(\Omega), \int_\Omega |\nabla u|^2 \le 1} \int_\Omega \left(e^{\alpha u^2} - 1\right) < +\infty \quad \textit{for some } \alpha > 0$$

3. *Poincaré inequality holds for Ω, that is*

$$\lambda_1(\Omega) = \inf_{0 \not\equiv u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} > 0$$

4. *Ω's inner radius*

$$\omega(\Omega) = \sup\{R > 0 : \exists x \in \Omega \text{ such that } B_R(x) \subset \Omega\}$$

is finite.

However, in the general case, things are not so simple: subcritical exponential integrability trivially implies Poincaré inequality, but the converse is no longer true; some original results have been given in this context.
Actually, if the metric goes to infinity faster than g_h around some point of $\partial\Omega$, then Poincaré inequality does not hold, as proved by Gabriele Mancini in his degree thesis, precisely:

Theorem 11.

Let $g_\rho = \rho(x) g_e = \zeta(x) g_h$ be a conformal metric defined on $\Omega = B_1(0) \subset \mathbb{R}^N$ such that

$$\lim_{x \to \widetilde{x}} \zeta(x) = +\infty \quad \text{for some } \widetilde{x} \in \partial\Omega$$

Then, $\lambda_1(\Omega, g_\rho) = 0$.

However, there exist some strange metrics where exponential integrability does not hold up to the exponent α_N but it does for lower ones:

Theorem 12.

Let $g_\rho = \rho(x) g_e = \zeta(x) g_h$ be a conformal metric on $\Omega = B_1(0) \subset \mathbb{R}^N$ such that

$$0 \leq \zeta - C \in L^q(\Omega, g_h) \quad \text{for some } q > 1, C > 0$$

Then,

$$\sup_{u \in W_0^{1,N}(\Omega, g_\rho), \int_\Omega |\nabla_{g_\rho} u|^N dV_{g_\rho} \leq 1} \int_\Omega \Phi(\theta u) dV_{g_\rho} < +\infty \quad \forall \theta < 1 - \frac{1}{q}$$

Moreover, for some of these metrics, the critical exponent

$$\widetilde{\alpha} = \sup \left\{ \alpha > 0 : \sup_{u \in W_0^{1,N}(\Omega, g_\rho), \int_\Omega |\nabla_{g_\rho} u|^N dV_{g_\rho} \leq 1} \int_\Omega \Phi\left(\frac{\alpha}{\alpha_N} u\right) < +\infty \right\}$$

is strictly less than α_N

Theorem 13.

For any $q > 1$, there exist some conformal metrics satisfying the hypotheses of theorem 12 and

$$\sup_{0 \not\equiv u \in W_0^{1,N}(\Omega, g_\rho), \int_\Omega |\nabla_{g_\rho} u|^N dV_{g_\rho} \leq 1} \int_\Omega \Phi(\theta u) dV_{g_\rho} = +\infty \quad \forall \theta > 1 - \frac{1}{Nq}$$

Finally, one can easily discover that, for any metric that verifies Poincaré inequality, sharp Moser-Trudinger inequality holds for radially decreasing functions; this is quite surprising, since for Euclidean domains and for hyperbolic metric one can apply symmetrization to find out that the supremum in Moser-Trudinger inequality is the same as the one taken between radially decreasing functions: the following result shows that this does not happen for general metrics.

Proposition 14.

Let $g_\rho = \rho(x) g_e$ be a conformal metric on $\Omega = B_1(0) \subset \mathbb{R}^N$ such that $\lambda_1(\Omega, g_\rho) > 0$. Then, setting

$$\widetilde{W}(\Omega, g_\rho) = \Big\{ 0 \leq u \in W_0^{1,N}(\Omega, g_\rho) \text{ radially nonincreasing}$$

$$\text{such that } \int_\Omega |\nabla_{g_\rho} u|^N dV_{g_\rho} \leq 1 \Big\}$$

one has

$$\sup_{u \in \widetilde{W}(\Omega, g_\rho)} \int_\Omega \Phi(u) dV_{g_\rho} < +\infty$$

Finally, one considers the case of unbounded Euclidean domains.
Again, it is very simple to notice that the positivity of the first eigenvalue of $-\Delta$ on Ω is a necessary condition for Moser-Trudinger inequality to hold; however, if the Sobolev norm $\int_\Omega \left(|u|^N + |\nabla u|^N\right)$ is set to be less than 1, the same result as Moser's holds for any domain, and moreover the supremum is bounded independently by Ω; this result was proved by Ruf [Ruf05] for $N = 2$ and later extended in any dimension by Li and Ruf himself [LR08].

Theorem 15.

Let $\Omega \subset \mathbb{R}^N$ be an open domain.
Then

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega (|u|^N + |\nabla u|^N) \leq 1} \int_\Omega \Phi(u) \leq C < +\infty$$

where $C = C(N)$ is a constant that depends only on the dimension.

This result allows to give a partial extension to theorem 10, that is Poincaré inequality implies Moser-Trudinger inequality for any subcritical exponent $\alpha < \alpha_N$.

Corollary 16.

Let $\Omega \subset \mathbb{R}^N$ be an open domain.
Then the following conditions are equivalent:

1. *Subcritical Moser-Trudinger inequality holds for Ω, that is*

$$\sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \leq 1} \int_\Omega \Phi(\theta u) < +\infty \quad \forall \theta \in (0,1)$$

2. Poincaré inequality holds for Ω, that is

$$\lambda_1(\Omega) = \inf_{0 \not\equiv u \in W_0^{1,N}(\Omega)} \frac{\int_\Omega |\nabla u|^N}{\int_\Omega |u|^N} > 0$$

A stronger result was recently given by Ga. Mancini, who proved that Poincaré inequality is actually equivalent to sharp Moser-Trudinged inequality for any domain $\Omega \subset \mathbb{R}^N$.

Extremal functions for Moser-Trudinger inequality

The first result concerning the problem of extremal functions for Moser-Trudinger inequality is the so-called concentration-compactness principle by Lions [Lio85]; whereas for subcritical exponent the Moser-Trudinger functional is compact, in the case $\alpha = \alpha_N$ there exist some noncompact sequences which concentrate in a point, but this is the only possible alternative to compactness:

Theorem 17.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and u_k a sequence satisfying $\int_\Omega |\nabla u_k|^N \leq 1$. Then, up to a subsequence,

$$u_k \underset{k\to+\infty}{\rightharpoonup} u \quad and \quad |\nabla u_k|^N dx \underset{k\to+\infty}{\rightharpoonup} \mu$$

for some $u \in W_0^{1,N}(\Omega)$ and a probability measure μ on $\overline{\Omega}$, and one of the following occurs:

1. *$u \equiv 0$ and $\mu = \delta_{\widetilde{x}}$ for some $\widetilde{x} \in \overline{\Omega}$, and*
$$\lim_{k\to+\infty} \int_{\Omega \setminus B_\varepsilon(\widetilde{x})} \left(e^{\alpha_N |u_k|^{\frac{N}{N-1}}} - 1 \right) = 0 \quad \forall \varepsilon > 0$$
2. *$e^{\alpha_N |u_k|^{\frac{N}{N-1}}}$ is bounded in $L^p(\Omega)$ for some $p > 1$ and converges in $L^1(\Omega)$ to $e^{\alpha_N |u|^{\frac{N}{N-1}}}$.*

Hence, to prove the existence of extremals, the following two steps suffice: estimating the maximum level of concentrating sequences and showing that the functional attains higher values; this is the method used by Carleson and Chang [CC86], who proved the first existence result of this kind, for balls.

Theorem 18.

Let $\Omega = B_R(\widetilde{x}) \subset \mathbb{R}^N$ *be the ball centered in* $\widetilde{x} \in \mathbb{R}^N$ *with radius equal to* $R > 0$.

Then, there exists a function $\widetilde{u} \in W_0^{1,N}(\Omega)$ *with* $\int_\Omega |\nabla \widetilde{u}|^N \leq 1$ *and*

$$\int_\Omega e^{\alpha_N |\widetilde{u}|^{\frac{N}{N-1}}} = \sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}}$$

This result was extended to bounded planar domains by Flucher [Flu92] and later generalized to any dimension by Lin [Lin96]; they show that the ratio between the supremum of Moser-Trudinger functional and the concentration level can only increase when passing from a ball to another domain, so it is always higher than 1 and so extremals are attained.

To prove this theorem, Flucher uses tools from complex analysis in the case of planar domains, and the Green function of $-\Delta$ for the general case; similarly, Lin uses the Green function of the N-Laplacian $-\Delta_N$ to extend Flucher's result.

Theorem 19.

Let $\Omega \subset \mathbb{R}^N$ *be a bounded domain.*

Then, setting

$$F_\Omega(u) = \int_\Omega \left(e^{\alpha_N |u|^{\frac{N}{N-1}}} - 1 \right)$$

and

$$C_\Omega(x) = \sup \left\{ \limsup_{k \to +\infty} F_\Omega(u_k) : u_k \text{ concentrates in } x \in \overline{\Omega} \right\}$$

it holds

$$\frac{\sup F_\Omega}{\sup C_\Omega} \geq \frac{\sup F_{B_1(0)}}{C_{B_1(0)}(0)} > 1$$

and there exists a function $\widetilde{u} \in W_0^{1,N}(\Omega)$ *with* $\int_\Omega |\nabla \widetilde{u}|^N \leq 1$ *and*

$$\int_\Omega e^{\alpha_N |\widetilde{u}|^{\frac{N}{N-1}}} = \sup_{u \in W_0^{1,N}(\Omega), \int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}}$$

The arguments used by Carleson-Chang, Flucher and Lin cannot be extended to search extremals in unbounded domains, since the concentration-compactness principle does not hold in this case, mainly because of noncompact sequences which vanish at the infinity.

Hence, in the last part of the thesis, where the existence of extremals for Moser-Trudinger inequality is studied for simply connected unbounded domains, one has to proceed differently: first of all, one considers domains which are symmetric with respect to two orthogonal axes, such as the strip $\Omega = \mathbb{R} \times (-1, 1)$, and applies one-dimensional symmetrization with respect to each axes; this operation shares some properties with the radial Schwarz symmetrization on balls and thus allows to consider for the supremum only functions with are even and decreasing with respect to both variables, which have good decay properties.
Moreover, one uses the conformal diffeomorphism between Ω and $B_1(0)$ to make calculations for maximizing sequences; concentration can be excluded, whereas for vanishing sequences, due to the decay properties mentioned before, all the terms of the Moser-Trudinger functional tend to 0 except for $4\pi u^2$, hence the maximum value is $\frac{4\pi}{\lambda_1(\Omega)} = \frac{16}{\pi}$; finally, some functions such that the Moser-Trudinger functional attains higher values than the last quantity are found, and thus the existence of extremals for Ω has been proved:

Theorem 20.
If $\Omega = \mathbb{R} \times (-1, 1) \subset \mathbb{R}^2$, then there exists a function $\widetilde{u} \in H_0^1(\Omega)$ such that $\int_\Omega |\nabla \widetilde{u}|^2 \leq 1$ and

$$\int_\Omega \left(e^{4\pi \widetilde{u}^2} - 1 \right) = \sup_{u \in H_0^1(\Omega), \int_\Omega |\nabla u|^2 \leq 1} \int_\Omega \left(e^{4\pi u^2} - 1 \right)$$

The existence of extremals for Moser-Trudinger inequality on the strip has been proved working with Ga. Mancini.

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