Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di Analisi 3

A.A. 2009-2010 - Docente: Prof. P. Esposito
Tutori: Gabriele Mancini, Luca Battaglia e Vincenzo Morinelli

Soluzioni del tutorato numero 4 (17 Marzo 2010)
Teoremi della funzione implicita e della funzione inversa

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
http://www.lifedreamers.it/liuck

1. $F(x_1,x_2,y)=e^{x_1x_2y}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\frac{1}{1+y}$.

   (a) F è di classe C^1 in un intorno dell'origine, inoltre $F(0,0,0)=0$ e $\frac{\partial F}{\partial y}(0,0,0)=x_1x_2e^{x_1x_2y}+\frac{1}{(1+y)^2}\Big|_{(x_1,x_2,y)=(0,0,0)}=1\neq 0$, dunque per il teorema della funzione implicita $\exists r,\rho>0$ e $g\in C^1(B_r((0,0)),B_\rho(0))$ tale che $F(x_1,x_2,g(x))\equiv 0\ \forall x\in B_r((0,0))$.

   (b) Supponendo $r,\rho\leq\frac{1}{2}$, si ha che $|F(x_1,x_2,0)|=\left|1+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\right.$ $\left.-1\right|\leq\left|1-\cos\left(x_2^2\right)\right|+\left|\frac{x_1}{1+x_1}\right|\leq\frac{x_2^4}{2}+\frac{x_1}{1-\frac{1}{2}}\leq\frac{r^4}{2}+2r\leq\frac{5}{2}r$, dunque posto $T=\frac{1}{\frac{\partial F}{\partial y}(0,0,0)}=1$ per avere $\sup_{x\in B_r(0)}|F(x_1,x_2,0)|\leq\frac{\rho}{2}=\frac{\rho}{2\|T\|}$ è sufficiente prendere $r=\frac{\rho}{5}$; inoltre, $\left|1-T\frac{\partial F}{\partial y}(x_1,x_2,y)\right|=|1-$ $-x_1x_2e^{x_1x_2y}-\frac{1}{(1+y)^2}\Big|\leq|x_1x_2|e^{x_1x_2y}+\frac{y^2+2|y|}{(1+y)^2}\leq\frac{3}{2}\left(x_1^2+x_2^2\right)+$ $+\frac{\rho^2+2\rho}{\left(1-\frac{1}{2}\right)^2}\leq\frac{3}{2}r^2+12\rho\leq\frac{3}{2}r+12\rho\leq\frac{123}{10}\rho$, dunque per avere $\sup_{(x_1,x_2,y)\in B_r((0,0))\times B_\rho(0)}\left|1-T\frac{\partial F}{\partial y}(x_1,x_2,y)\right|\leq\frac{1}{2}$ è sufficiente prendere $\rho=\frac{5}{123}$, e di conseguenza $r=\frac{1}{123}$.

   (c) Essendo $e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\frac{1}{1+g(x_1,x_2)}\equiv 0\ \forall x\in$ $\in B_r(0,0)$, allora $0=\frac{d}{dx_1}\left(e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\right.$ $\left.-\frac{1}{1+g(x_1,x_2)}\right)\Big|_{(x_1,x_2)=(0,0)}=\left(x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)\right)e^{x_1x_2g(x_1,x_2)}+$ $+\frac{1}{(1+x_1)^2}+\frac{\frac{\partial g}{\partial x_1}(x_1,x_2)}{(1+g(x_1,x_2))^2}\Big|_{(x_1,x_2)=(0,0)}=1+\frac{\partial g}{\partial x_1}(0,0)\Rightarrow\frac{\partial g}{\partial x_1}(0,0)=$ $=-1$, analogamente $0=\frac{d}{dx_2}\left(e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\right.$ $\left.-\frac{1}{1+g(x_1,x_2)}\right)\Big|_{(x_1,x_2)=(0,0)}=\left(x_1g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)\right)e^{x_1x_2g(x_1,x_2)}-$

$$-2x_2\sin\left(x_2^2\right)+\left.\frac{\frac{\partial g}{\partial x_2}(x_1,x_2)}{(1+g(x_1,x_2))^2}\right|_{(x_1,x_2)=(0,0)}=\frac{\partial g}{\partial x_2}(0,0)\Rightarrow\frac{\partial g}{\partial x_2}(0,0)=$$

$$=0,\ \frac{d^2}{dx_1^2}\left(e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\frac{1}{1+g(x_1,x_2)}\right)\Bigg|_{(x_1,x_2)=(0,0)}=$$

$$=\left(\left(x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)\right)^2+2x_2\frac{\partial g}{\partial x_1}(x_1,x_2)+\right.$$

$$\left.+x_1x_2\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)\right)e^{x_1x_2g(x_1,x_2)}-\frac{2}{(1+x_1)^3}+\frac{\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)(1+g(x_1,x_2))^2}{(1+g(x_1,x_2))^4}-$$

$$-\left.\frac{2\left(\frac{\partial g}{\partial x_1}(x_1,x_2)\right)^2(1+g(x_1,x_2))}{(1+g(x_1,x_2))^4}\right|_{(x_1,x_2)=(0,0)}=\frac{\partial^2 g}{\partial x_1^2}(0,0)-4\Rightarrow$$

$$\Rightarrow\frac{\partial^2 g}{\partial x_1^2}(0,0)=4,\ 0=\frac{d^2}{dx_1dx_2}\left(e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\right.$$

$$\left.-\frac{1}{1+g(x_1,x_2)}\right)\Bigg|_{(x_1,x_2)=(0,0)}=\left(\left(g(x_1,x_2)+x_2\frac{\partial g}{\partial x_2}(x_1,x_2)+\right.\right.$$

$$\left.+x_1\frac{\partial g}{\partial x_1}(x_1,x_2)+x_1x_2\frac{\partial^2 g}{\partial x_1\partial x_2}\right)+\left(x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)\right)(x_1g(x_1,x_2)+$$

$$\left.\left.+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)\right)\right)e^{x_1x_2g(x_1,x_2)}+\frac{\frac{\partial^2 g}{\partial x_1\partial x_2}(1+g(x_1,x_2))^2}{(1+g(x_1,x_2))^4}-$$

$$-\left.\frac{2\frac{\partial g}{\partial x_1}(x_1,x_2)\frac{\partial g}{\partial x_2}(x_1,x_2)(1+g(x_1,x_2))}{(1+g(x_1,x_2))^4}\right|_{(x_1,x_2)=(0,0)}=\frac{\partial^2 g}{\partial x_1\partial x_2}(0,0)\Rightarrow$$

$$\Rightarrow\frac{\partial^2 g}{\partial x_1\partial x_2}(0,0)=0\text{ e }\frac{d^2}{dx_2^2}\left(e^{x_1x_2g(x_1,x_2)}+\cos\left(x_2^2\right)-\frac{1}{1+x_1}-\right.$$

$$\left.-\frac{1}{1+g(x_1,x_2)}\right)\Bigg|_{(x_1,x_2)=(0,0)}=\left(\left(x_1g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)\right)^2+\right.$$

$$\left.+2x_1\frac{\partial g}{\partial x_1}(x_1,x_2)+x_1x_2\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)\right)e^{x_1x_2g(x_1,x_2)}-2\sin\left(x_2^2\right)-$$

$$-4x_2^2\cos\left(x_2^2\right)+\left.\frac{\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)(1+g(x_1,x_2))^2-2\left(\frac{\partial g}{\partial x_2}(x_1,x_2)\right)^2(1+g(x_1,x_2))}{(1+g(x_1,x_2))^4}\right|_{(x_1,x_2)=(0,0)}=$$

$$=\frac{\partial^2 g}{\partial x_2^2}(0,0)\Rightarrow\frac{\partial^2 g}{\partial x_2^2}(0,0)=0\Rightarrow g(x_1,x_2)=g(0,0)+\langle\nabla g(0,0),(x_1,x_2)\rangle+$$

$$+\frac{\langle H_g(0,0)(x_1,x_2),(x_1,x_2)\rangle}{2}+o\left(x_1^2+x_2^2\right)=1-x_1+2x_1^2+o\left(x_1^2+\right.$$

$$\left.+x_2^2\right).$$

2. $F(x,y_1,y_2)=\left(\log y_2+e^{y_1}-\cos x,\arctan(y_1y_2)-\dfrac{\sin x}{2+y_1^2}\right).$

   (a) F è di classe C^1 in un intorno del punto $(0,0,1)$, inoltre $F(0,0,1)=(0,0)$ e $\dfrac{\partial F}{\partial y}(0,0,1)=\left.\begin{pmatrix} e^{y_1} & \frac{1}{y_2}\\ \frac{y_2}{1+y_1^2y_2^2}-\frac{2y_1\sin x}{\left(2+y_1^2\right)^2} & \frac{y_1}{1+y_1^2y_2^2}\end{pmatrix}\right|_{(x,y_1,y_2)=(0,0,1)}=$

$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, è invertibile (con $T = \left(\dfrac{\partial F}{\partial y}(0,0,1)\right)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$), dunque per il teorema della funzione implicita $\exists r, \rho > 0$ e $g \in$ $\in C^1(B_r(0), B_\rho((0,1)))$ tale che $F(x, g_1(x), g_2(x)) \equiv 0 \; \forall x \in B_r(0)$.

(b) Supponendo $r \le 1$ e $\rho \le \dfrac{1}{2}$, si ha che $\|F(x,0,1)\| = \sqrt{(1-\cos x)^2 + \dfrac{\sin^2 x}{4}} \le$ $\le \sqrt{\dfrac{x^4}{4} + \dfrac{x^2}{4}} \le \sqrt{\dfrac{r^4}{4} + \dfrac{r^2}{2}} \le \dfrac{r}{\sqrt{2}} \le \dfrac{3}{4} r$, dunque per avere $\sup\limits_{x \in B_r(0)} \|F(x,0,1)\| \le$ $\le \dfrac{\rho}{4} = \dfrac{\rho}{4\|T\|_\infty} \le \dfrac{\rho}{2\|T\|}$ è sufficiente prendere $r = \dfrac{\rho}{3}$; inoltre, $\mathbb{I}_2 -$ $-T \dfrac{\partial F}{\partial y}(x, y_1, y_2) = \begin{pmatrix} 1 - \frac{y_2}{1+y_1^2 y_2^2} - \frac{2y_1 \sin x}{(2+y_1^2)^2} & -\frac{y_1}{1+y_1^2 y_2^2} \\ \frac{y_2}{1+y_1^2 y_2^2} + \frac{2y_1 \sin x}{(2+y_1^2)^2} - e^{y_1} & 1 - \frac{1}{y_2} + \frac{y_1}{1+y_1^2 y_2^2} \end{pmatrix}$, dunque poiché $\left|1 - \dfrac{y_2}{1+y_1^2 y_2^2} - \dfrac{2y_1 \sin x}{(2+y_1^2)^2}\right| \le \dfrac{|1-y_2| + y_1^2 y_2^2}{1+y_1^2 y_2^2} + \dfrac{2|y_1||\sin x|}{(2+y_1^2)^2} \le$ $\le |1-y_2| + y_1^2 \dfrac{9}{4} + \dfrac{2|y_1||x|}{4} \le \rho + \dfrac{9}{4}\rho^2 + \dfrac{r\rho}{2} \le \rho + \dfrac{9}{4}\rho + \dfrac{\rho^2}{6} = \dfrac{41}{12}\rho$, $\left|-\dfrac{y_1}{1+y_1^2 y_2^2}\right| \le |y_1| \le \rho$, $\left|\dfrac{y_2}{1+y_1^2 y_2^2} - \dfrac{2y_1 \sin x}{(2+y_1^2)^2} - e^{y_1}\right| \le \dfrac{|1-y_2| + y_1^2 y_2^2}{1+y_1^2 y_2^2} +$ $+ \dfrac{2|y_1||\sin x|}{(2+y_1^2)^2} + |1 - e^{y_1}| \le |1-y_2| + y_1^2 \dfrac{9}{4} + \dfrac{|y_1||x|}{2} + 3|y_1| \le \rho + \dfrac{9}{4}\rho^2 +$ $+ \dfrac{r\rho}{2} + 3\rho \le 4\rho + \dfrac{9}{4}\rho + \dfrac{\rho^2}{6} \le \dfrac{77}{12}\rho$ e $\left|1 - \dfrac{1}{y_2} + \dfrac{y_1}{1+y_1^2 y_2^2}\right| \le \dfrac{|y_2 - 1|}{|y_2|} +$ $+ |y_1| \le 2|y_2 - 1| + |y_1| \le 3\rho$, dunque per avere $\sup\limits_{(x,y_1,y_2) \in B_r(0) \times B_\rho(0,0)} \Big\| \mathbb{I}_2 -$ $T \dfrac{\partial F}{\partial y}(x, y_1, y_2) \Big\| \le 2 \sup\limits_{(x,y_1,y_2) \in B_r(0) \times B_\rho(0,0)} \left\| \mathbb{I}_2 - T \dfrac{\partial F}{\partial y}(x, y_1, y_2) \right\|_\infty \le$ $\le \dfrac{1}{2}$ è sufficiente prendere $\rho = \dfrac{3}{77}$, e di conseguenza $r = \dfrac{1}{77}$.

(c) Essendo $\log(g_2(x)) + e^{g_1(x)} - \cos x \equiv 0 \; \forall x \in B_r(0)$, allora $0 =$ $= \dfrac{d}{dx}\left(\log(g_2(x)) + e^{g_1(x)} - \cos x\right)\Big|_{x=0} = \Big(g_1'(x) e^{g_1(x)} + \dfrac{g_2'(x)}{g_2(x)} +$ $+ \sin x \Big)\Big|_{x=0} = g_1'(0) + g_2'(0) \Rightarrow g_1'(0) = -g_2'(0)$; analogamente, essendo $0 = \dfrac{d}{dx} \arctan(g_1(x) g_2(x)) + \dfrac{\sin x}{2 + g_1^2(x)}\Big|_{x=0} = \dfrac{g_1(x) g_1'(x) + g_2(x) g_2'(x)}{1 + g_1^2(x) g_2^2(x)} -$ $- \dfrac{\cos x}{2 + g_1^2(x)} + \dfrac{2 \sin x \, g_1(x) g_1'(x)}{(2 + g_1^2(x))}\Big|_{x=0} = g_1'(0) - \dfrac{1}{2} \Rightarrow g_1'(0) = \dfrac{1}{2} \Rightarrow$ $\Rightarrow g_2'(0) = -\dfrac{1}{2} \Rightarrow g_1(x) = \dfrac{x}{2} + o(x)$ e $g_2(x) = 1 - \dfrac{x}{2} + o(x)$.

3. Sia $F : \mathbb{R} \to \mathbb{R}$ definita da $F(y) = y^2 + y + \cosh y$.

(a) F è di classe C^1 in un intorno dell'origine con $F(0) = 1$, inoltre $F'(0) = 2y + 1 +$ $+ \sinh y|_{y=0} = 1 \ne 0$, dunque per il teorema della funzione inversa $\exists r, \rho > 0$ e $g \in C^1(B_r(1), B_\rho(0))$ tale che $F(g(u)) = u \; \forall u \in B_r(1)$.

(b) Supponendo $\rho \le 1$, posto $T = \dfrac{1}{F'(0)} = 1$ si ha che $|1 - TF'(y)| = |-2y| +$

$+|\sinh y| \leq 2|y| + |\sinh y| \leq 5|y| \leq 5\rho$, dunque affinché $\sup_{y\in B_r(0)} |1-$
$-TF'(y)| = 5\rho \leq \frac{1}{2}$ è sufficiente prendere $\rho = \frac{1}{10}$ e $r = \frac{\rho}{2|T|} = \frac{\rho}{2} =$
$= \frac{1}{20}$.

(c) Essendo $F(g(u)) = u\ \forall u \in B_r(1)$, allora $1 = \frac{d}{du}F(g(u)) = F'(g(u))g'(u)$ $\forall u \in B_r(1) \Rightarrow g'(1) = \frac{1}{F'(0)} = 1$ e analogamente $0 = \frac{d^2}{du^2}F(g(u)) =$ $= F''(g(u))g'(u)^2 + F'(g(u))g''(u) \Rightarrow g''(1) = -\frac{F''(0)g'(1)^2}{F'(0)} = -3$, perché $F''(0) = 2 + \cosh y|_{y=0} = 3$, dunque $g(u) = g(1) + g'(1)(u-1) +$ $\frac{g''(1)}{2}(u-1)^2 + o\left((u-1)^2\right) = (u-1) - \frac{3}{2}(u-1)^2 + o\left((u-1)^2\right)$.

4. $F(x,y) = \left(\sqrt{1+x^2} - e^{\arctan y}, x + x^3 + y^2\right)$.

(a) F è di classe C^1 in un intorno dell'origine con $F(0,0) = (0,0)$, inoltre $\frac{\partial F}{\partial(x,y)}(0,0) = \left.\begin{pmatrix} \frac{x}{\sqrt{1+x^2}} & -\frac{e^{\arctan y}}{1+y^2} \\ 1+3x^2 & 2y \end{pmatrix}\right|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ è invertibile (con $T = \left(\frac{\partial F}{\partial(x,y)}(0,0)\right)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$), dunque per il teorema della funzione inversa $\exists r, \rho > 0$ e $g \in C^1(B_r((0,0)), B_\rho((0,0)))$ tale che $F(g(u,v)) = (u,v)\ \forall (u,v) \in B_r((0,0))$.

(b) Supponendo $\rho \leq 1$, si ha che $\mathbb{I}_2 - T\frac{\partial F}{\partial(x,y)}(0,0) = \begin{pmatrix} -3x^2 & -2y \\ \frac{x}{\sqrt{1+x^2}} & 1 - \frac{e^{\arctan y}}{1+y^2} \end{pmatrix}$, dunque poiché $\left|-3x^2\right| \leq 3\rho^2 \leq 3\rho$, $|-2y| \leq 2\rho$, $\left|\frac{x}{\sqrt{1+x^2}}\right| \leq 2|x| \leq 2\rho$ e $\left|1 - \frac{e^{\arctan y}}{1+y^2}\right| \leq \frac{y^2}{1+y^2} + \frac{|1-e^{\arctan y}|}{1+y^2} \leq y^2 + \left|1 - e^{\arctan y}\right| \leq \rho^2 +$ $+3|y| \leq 4\rho$, dunque per avere $\sup_{(x,y)\in B_\rho((0,0))} \left\|\mathbb{I}_2 - T\frac{\partial F}{\partial(x,y)}(x,y)\right\| \leq$ $\leq 2 \sup_{(x,y)\in B_\rho((0,0))} \left\|\mathbb{I}_2 - T\frac{\partial F}{\partial(x,y)}(x,y)\right\|_\infty \leq \frac{1}{2}$ è sufficiente prendere $\rho = \frac{1}{16}$, e di conseguenza $r = \frac{1}{64} = \frac{\rho}{4\|T\|_\infty} \leq \frac{\rho}{2\|T\|}$.

(c) Essendo $F(g(u,v)) = (u,v)\ \forall (u,v) \in B_r((0,0))$, allora $\frac{\partial F}{\partial(x,y)}(g(u,v))\frac{\partial g}{\partial(u,v)}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \frac{\partial g}{\partial(u,v)}(0,0) =$ $= \left(\frac{\partial F}{\partial(x,y)}(0,0)\right)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow g_1(u,v) = g_1(0,0) +$ $+ \left\langle \left(\frac{\partial g_1}{\partial u}(0,0), \frac{\partial g_1}{\partial v}(0,0)\right), (u,v) \right\rangle + o\left(\sqrt{u^2+v^2}\right) = v + o\left(\sqrt{u^2+v^2}\right)$ e analogamente $g_2(u,v) = -u + o\left(\sqrt{u^2+v^2}\right)$.

5. $F(x,y) = \left(\arctan x + \dfrac{x^2}{2} - \ln\cos y, y + \dfrac{e^{xy}}{1+y^2} + \cosh\left(x^2+y^2\right)\right)$.

(a) F è di classe C^1 in un intorno dell'origine con $F(0,0) = (0,2)$, inoltre

$$\frac{\partial F}{\partial(x,y)}(0,0) = \left.\begin{pmatrix} \frac{1}{1+x^2} + x & \tan y \\ \frac{ye^{xy}}{1+y^2} + 2x\sinh\left(x^2+y^2\right) & 1 + \frac{xe^{xy}\left(1+y^2\right) - 2ye^{xy}}{\left(1+y^2\right)^2} + 2y\sinh\left(x^2+y^2\right) \end{pmatrix}\right|_{(x,y)=(0,0)}$$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ è invertibile (con $T = \left(\dfrac{\partial F}{\partial y}(0,0)\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), dunque per il teorema della funzione inversa $\exists r, \rho > 0$ e $g \in C^1(B_r((0,0)), B_\rho((0,0)))$ tale che $F(g(u,v)) = (u,v)\ \forall (u,v) \in B_r((0,0))$.

(b) Supponendo $\rho \leq 1$, si ha che $\mathbb{I}_2 - T\dfrac{\partial F}{\partial(x,y)} =$

$$= \begin{pmatrix} 1 - \frac{1}{1+x^2} - x & -\tan y \\ -\frac{ye^{xy}}{1+y^2} - 2x\sinh\left(x^2+y^2\right) & -\frac{xe^{xy}\left(1+y^2\right) - 2ye^{xy}}{\left(1+y^2\right)^2} - 2y\sinh\left(x^2+y^2\right) \end{pmatrix},$$

dunque poiché $\left|1 - \dfrac{1}{1+x^2} - x\right| \leq \dfrac{x^2}{1+x^2} + |x| \leq x^2 + |x| \leq \rho^2 + \rho \leq$ $\leq 2\rho$, $|-\tan y| \leq 2|y| \leq 2\rho$, $\left|-\dfrac{ye^{xy}}{1+y^2} - 2x\sinh\left(x^2+y^2\right)\right| \leq |y|e^{xy} +$ $+ 2|x|\sinh\left(x^2+y^2\right) \leq 3|y| + 6|x|\left(x^2+y^2\right) \leq 3\rho + 6\rho^3 \leq 6\rho$ e $\left|-\dfrac{xe^{xy}\left(1+y^2\right) - 2ye^{xy}}{\left(1+y^2\right)^2} - 2y\sinh\left(x^2+y^2\right)\right| \leq |x|e^{xy} + 2|y|e^{xy} +$ $+ 2|y|\sinh\left(x^2+y^2\right) \leq 3|x| + 6|y| + 6|y|\left(x^2+y^2\right) \leq 9\rho + 6\rho^3 \leq 15\rho$, dunque per avere $\sup\limits_{(x,y)\in B_\rho((0,0))} \left\|\mathbb{I}_2 - T\dfrac{\partial F}{\partial(x,y)}(x,y)\right\| \leq 2\left\|\mathbb{I}_2 - T\dfrac{\partial F}{\partial(x,y)}(x,y)\right\|_\infty \leq$ $\leq \dfrac{1}{2}$ è sufficiente prendere $\rho = \dfrac{1}{30}$, e di conseguenza $r = \dfrac{1}{120} = \dfrac{\rho}{4\|T\|_\infty} \leq$ $\leq \dfrac{\rho}{2\|T\|}$.

(c) Essendo $F(g(u,v)) = (u,v)\ \forall (u,v) \in B_r((0,2))$, allora $\dfrac{\partial F}{\partial(x,y)}(g(u,v))\dfrac{\partial g}{\partial(u,v)}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \dfrac{\partial g}{\partial(u,v)}(0,2) =$ $\left(\dfrac{\partial F}{\partial(x,y)}(0,0)\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, inoltre $0 = \left.\dfrac{d^2}{du^2}u\right|_{(u,v)=(0,2)} =$

$$= \left.\frac{d^2}{du^2}\left(\arctan(g_1(u,v)) + \frac{g_1^2(u,v)}{2} - \ln\cos(g_2(u,v))\right)\right|_{(u,v)=(0,2)} =$$

$$= \frac{\frac{\partial^2 g_1}{\partial u^2}(u,v)\left(1+g_1^2(u,v)\right)}{\left(1+g_1^2(u,v)\right)^2} - \frac{2g_1(u,v)\left(\frac{\partial g_1}{\partial u}(u,v)\right)^2}{\left(1+g_1^2(u,v)\right)^2} + g_1(u,v)\frac{\partial^2 g_1}{\partial u^2}(u,v) +$$

$$\left. + \left(\frac{\partial g_1}{\partial u}(u,v)\right)^2 + \frac{\left(\frac{\partial g_2}{\partial u}(u,v)\right)^2}{\cos^2(g_2(u,v))} + \tan(g_2(u,v))\frac{\partial^2 g_2}{\partial u^2}(u,v)\right|_{(u,v)=(0,2)} =$$

$= \dfrac{\partial^2 g_1}{\partial u^2}(0,2) + 1 \Rightarrow \dfrac{\partial^2 g_1}{\partial u^2}(0,2) = -1$, analogamente $0 = \left.\dfrac{d^2}{dudv}u\right|_{(u,v)=(0,2)} =$

$$= \left.\frac{d^2}{dudv}\left(\arctan(g_1(u,v)) + \frac{g_1^2(u,v)}{2} - \ln\cos(g_2(u,v))\right)\right|_{(u,v)=(0,2)} =$$

$$\frac{\frac{\partial^2 g_1}{\partial u\partial v}(u,v)\left(1+g_1^2(u,v)\right)}{\left(1+g_1^2(u,v)\right)^2} - \frac{2g_1(u,v)\frac{\partial g_1}{\partial u}(u,v)\frac{\partial g_1}{\partial v}(u,v)}{\left(1+g_1^2(u,v)\right)^2} + g_1(u,v)\frac{\partial^2 g_1}{\partial u\partial v}(u,v)+$$

$$+\frac{\partial g_1}{\partial u}(u,v)\frac{\partial g_1}{\partial v}(u,v) - \frac{\frac{\partial g_2}{\partial u}(u,v)\frac{\partial g_2}{\partial v}(u,v)}{\cos^2(g_2(u,v))} + \tan(g_2(u,v))\frac{\partial^2 g_2}{\partial u\partial v}(u,v)\Bigg|_{(u,v)=(0,2)} =$$

$$= \frac{\partial^2 g_1}{\partial u\partial v}(0,2) \Rightarrow \frac{\partial^2 g_1}{\partial u^2}(0,2) = 0 \text{ e } 0 = \frac{d^2}{dv^2}u\Bigg|_{(u,v)=(0,2)} = \frac{d^2}{dv^2}\left(\arctan(g_1(u,v))+\right.$$

$$\left.+\frac{g_1^2(u,v)}{2} - \ln\cos(g_2(u,v))\right)\Bigg|_{(u,v)=(0,2)} = \frac{\frac{\partial^2 g_1}{\partial v^2}(u,v)\left(1+g_1^2(u,v)\right)}{\left(1+g_1^2(u,v)\right)^2} -$$

$$-\frac{2g_1(u,v)\left(\frac{\partial g_1}{\partial v}(u,v)\right)^2}{\left(1+g_1^2(u,v)\right)^2} + g_1(u,v)\frac{\partial^2 g_1}{\partial v^2}(u,v) + \left(\frac{\partial g_1}{\partial v}(u,v)\right)^2 +$$

$$+\frac{\left(\frac{\partial g_2}{\partial v}(u,v)\right)^2}{\cos^2(g_2(u,v))} + \tan(g_2(u,v))\frac{\partial^2 g_2}{\partial v^2}(u,v)\Bigg|_{(u,v)=(0,2)} = \frac{\partial^2 g_1}{\partial v^2}(0,2)+$$

$$+1 \Rightarrow \frac{\partial^2 g_1}{\partial u^2}(0,2) = -1 \Rightarrow g_1(u,v) = g_1(0,2) + \left\langle \left(\frac{\partial g_1}{\partial u}(0,2), \frac{\partial g_1}{\partial v}(0,2)\right), (u, v-2)\right\rangle +$$

$$+\frac{\left\langle \begin{pmatrix} \frac{\partial^2 g_1}{\partial u^2} & \frac{\partial^2 g_1}{\partial u\partial v} \\ \frac{\partial^2 g_1}{\partial u\partial v} & \frac{\partial^2 g_1}{\partial v^2}\end{pmatrix}(u, v-2), (u, v-2)\right\rangle}{2} + o\left(u^2+v^2\right) = u - \frac{u^2}{2} -$$

$$-\frac{(v-2)^2}{2} + o\left(u^2+(v-2)^2\right).$$