AM5: Tracce delle lezioni- XI Settimana

LA DISEGUAGLIANZA DI SOBOLEV

$$\forall p \in (1, N), \exists c = c(N,p) : \quad \left(\int_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} \leq c \int_{\mathbf{R}^N} |\nabla u|^p \qquad \forall u \in C_0^\infty(\mathbf{R}^N)$$

Una formula di rappresentazione. Sia $c_N := N \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}$. É

$$u(x) = \frac{1}{c_N} \int_{\mathbf{R}^N} \frac{< \nabla u(y)\, ,\, x-y >}{|x-y|^N}\, dy \qquad \forall u \in C_0^\infty(\mathbf{R}^N)$$

Tale formula si basa sulla **formula di integrazione per parti**

$$\int_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\, v = - \int_{\mathbf{R}^N} u\, \frac{\partial v}{\partial x_j} \qquad \forall u \in C^\infty, \quad \forall v \in C_0^\infty(\mathbf{R}^N)$$

che é a sua volta conseguenza del Teorema Fondamentale del Calcolo. Ad esempio,

$$\int_{\mathbf{R}^N} \frac{\partial (u\,v)}{\partial x_1} = \int_{\mathbf{R}^{N-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial (u\,v)}{\partial x_1} \right) dx_2 \dots dx_N = 0$$

Prova della formula di rappresentazione. Per ogni fissato x,

$$\int_{\mathbf{R}^N} \frac{< \nabla u(y)\, ,\, x-y >}{|x-y|^N}\, dy = \lim_{\epsilon \to 0} \sum_{j=1}^N \int_{\mathbf{R}^N} \frac{\partial u}{\partial y_j} \frac{x_j - y_j}{(\epsilon^2 + |x-y|^2)^{\frac{N}{2}}}\, dy =$$

$$- \lim_{\epsilon \to 0} \sum_{j=1}^N \int_{\mathbf{R}^N} u(y) \left(- \frac{1}{(\epsilon^2 + |x-y|^2)^{\frac{N}{2}}} + \frac{N}{2} \frac{2(x_j - y_j)^2}{(\epsilon^2 + |x-y|^2)^{\frac{N+2}{2}}} \right) =$$

$$N \lim_{\epsilon \to 0} \epsilon^2 \int_{\mathbf{R}^N} \left[\frac{u(y)}{(\epsilon^2 + |x-y|^2)^{\frac{N+2}{2}}} \right] dy =$$

$$= N \lim_{\epsilon \to 0} \int_{\mathbf{R}^N} \left[\frac{u(x - \epsilon z))}{(1+|z|^2)^{\frac{N+2}{2}}} \right] dy = N u(x) \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}$$

Prova della diseguaglianza di Sobolev. $u \in C_0^\infty(\mathbf{R}^N) \;\Rightarrow$

$$|u(x)| \;\leq\; c \int_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\; dy \;=\; c\; (|\nabla u| \, * G_{N-1})\,(x) \quad \forall x \in \mathbf{R}^N \;\Rightarrow$$

$$\|u\|_{\frac{Np}{N-p}} \;\leq\; c\, \|G_{N-1} * |\nabla u|\|_{\frac{Np}{N-p}} \;\leq\; c\; \|\nabla u\|_p$$

Diseguaglianza di POINCARÉ. Sia $1 \; < p < N$.

$\forall\, R \;>\; 0, \quad \exists\, c = c(N,p,R):$ $$\int_{\mathbf{R}^N} |\nabla u|^p \;\geq\; c \int_{\mathbf{R}^N} |u|^p \quad \forall u \in C_0^\infty(B_R)$$

Infatti, se $\frac{1}{q} + \frac{N-p}{N} = 1,$ allora, usando Holder e quindi Sobolev,

$$\int_{\mathbf{R}^N} |u|^p \;\leq\; c(N,p) \left(\int_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} vol(B_R)^{\frac{1}{q}} \;\leq\; c(N,p,R) \int_{\mathbf{R}^N} |\nabla u|^p$$

Diseguaglianze di MORREY. Sia $p \;>\; N$.

(i) $\forall\, R \;>\; 0 \;\exists\; c = c(N,p,R) \;:$ $\quad \|u\|_\infty \;\leq\; c \left(\int_{\mathbf{R}^N} |\nabla u|^p \right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(B_R)$

(ii) $\exists c = c(p,N):$ $\quad |u(x) - u(y)| \;\leq\; c|x-y|^{\frac{p-N}{p}} \left(\int_{\mathbf{R}^N} |\nabla u|^p \right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(\mathbf{R}^N)$

(i) Utilizzando la formula di rappresentazione e quindi Holder, ed usando il fatto che

$$p > N \;\Rightarrow\; \frac{1}{q} = 1 - \frac{1}{p} \;>\; \frac{N-1}{N} \;\Rightarrow\; q(N-1) \;<\; N$$

vediamo che

$$u \in C_0^\infty(B_R), \quad x \in \mathbf{R}^N \;\Rightarrow\; |u(x)| \;\leq\; c \int_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\; dy \;\leq$$

$$c\left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{B_R} \frac{1}{|x-y|^{q(N-1)}}\,dy\right)^{\frac{1}{q}} \le c\left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{B_{2R}} \frac{dz}{|z|^{q(N-1)}}\right)^{\frac{1}{q}}$$

(ii) Sia $Q_r := \{x : \ |x_i| \le r \ \forall i\}$ (cubo di lato $2r$ centrato nell'origine). Fissato $\overline{x}$, sia $\overline{u} = \frac{1}{2^N r^N} \int_{Q_r+\overline{x}} u$ la media di u su $Q := Q_r + \overline{x}$. Per ogni $x \in Q$ risulta

$$|\overline{u} - u(x)| = |\frac{1}{(2r)^N} \int_Q [u(y) - u(x)]\,dy| \le \int_Q \left[\frac{|y-x|}{(2r)^N} \int_0^1 |\nabla u(ty + (1-t)x)|\,dt\right] dy \le$$

$$\frac{\sqrt{N}}{(2r)^{N-1}} \int_0^1 \left(\int_{(1-t)x+tQ} \frac{|\nabla u(z)|}{t^N}\,dz\right) dt \le \frac{\sqrt{N}}{(2r)^{N-1}} \left(\int_Q |\nabla u|^p\right)^{\frac{1}{p}} \int_0^1 vol(tQ)^{1-\frac{1}{p}} \frac{dt}{t^N} =$$

$$\sqrt{N}\,(2r)^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}} \int_0^1 t^{-\frac{N}{p}}\,dt = c(N,p)\,r^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}}$$

Dunque, fissati x, y e posto $r = |x-y|$, $\overline{x} = \frac{x+y}{2}$, per cui $x, y \in Q_r + \overline{x}$, si ha

$$|u(x) - u(y)| \le 2c(N,p)\,r^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}} = 2c(N,p)|x-y|^{1-\frac{N}{p}} \left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}$$

Il Teorema di compattezza di RELLICH.

Sia $u_n \in C_0^\infty(B_R)$, con $\sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^p\right)^{\frac{1}{p}} < +\infty$. Allora

(i) se $1 < p < N$, u_n ha una sottosuccessione convergente in $L^r(B_R)\ \forall r < \frac{Np}{N-p}$.

(ii) se $p = N$, u_n ha una sottosuccessione convergente in $L^r(B_R)\ \ \forall r$.

(iii) se $p > N$, u_n ha una sottosuccessione uniformemente convergente in B_R

Prova. (i) Sia $1 \le r \le \frac{Np}{N-p}$. Da Holder e quindi Sobolev segue che

$$\sup_n \left(\int_{B_R} |u_n|^r \right)^{\frac{1}{r}} \le c(R) \sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^p \right)^{\frac{1}{p}} < +\infty$$

Sia ora $1 \le r < \frac{Np}{N-p}$. La diseguaglianza di interpolazione dá

$$\alpha \in [0,1),\ \alpha + (1-\alpha)\frac{N-p}{Np} = \frac{1}{r} : \quad \left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^r \, dx \right)^{\frac{1}{r}} \le$$

$$\left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)| \right)^{\alpha} \left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^{\frac{Np}{N-p}} \right)^{\frac{(1-\alpha)(N-p)}{Np}}$$

Il secondo fattore, grazie a Sobolev, resta, nelle nostre ipotesi, limitato e

$$vol(B_R)^{\frac{1}{p}-1} \int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)| dx \le vol(B_R)^{\frac{1}{p}-1} \int_{\mathbf{R}^N} \left(\int_0^1 | < \nabla u_n(x+th), h > | dt \right) dx$$

$$\le \ |h| \int_0^1 \left(\int_{\mathbf{R}^N} |\nabla u_n(x+th)|^p \, dx \right)^{\frac{1}{p}} dt \ = |h| \left(\int_{\mathbf{R}^N} |\nabla u_n(x)|^p \, dx \right)^{\frac{1}{p}} \le c\,|h|$$

La compattezza di u_n in $L^r(\mathbf{R}^N)$ segue quindi da Frechet-Kolmogoroff.

(ii) In tal caso $\sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^r \right)^{\frac{1}{r}} < +\infty \ \ \forall r$, e quindi, come in (i), otteniamo la compattezza di u_n in ogni L^r.

(iii) La (i) nel Teorema di Morrey dice $\sup_n \|u_n\|_\infty < +\infty$ mentre la (ii) assicura la equicontinuitá delle u_n. La conclusione segue quindi dal Teorema di Ascoli-Arzelá.

Poincaré e Morrey (i) non valgono in $\mathbf{R}^N$. Se $u_\epsilon(x) := \epsilon^{\frac{N}{p}} u(\epsilon x)$, é $\int_{\mathbf{R}^N} |u_\epsilon|^p = 1$ mentre $\int_{\mathbf{R}^N} |\nabla u_\epsilon|^p \to_{\epsilon\to 0} 0$; se $p > N$ e $u_\epsilon(x) := \epsilon^{\frac{N}{p}-1} u(\epsilon x)$, é $\int_{\mathbf{R}^N} |\nabla u_\epsilon|^p = 1$ mentre $\|u_\epsilon\|_\infty \ge \epsilon^{\frac{N}{p}-1} |u(0)| \to_{\epsilon\to 0} +\infty$ se $u(0) \ne 0$.

Analogamente si vede che **Rellich non vale in tutto $\mathbf{R}^N$ né fino all'esponente limite** $p^* := \frac{Np}{N-p}$.

SPAZI DI SOBOLEV: $H^1(\mathbf{R}^N)$ e $\mathcal{D}^1(\mathbf{R}^N)$

Abbiamo visto che

$$\int_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\; v \;=\; -\int_{\mathbf{R}^N} u\; \frac{\partial v}{\partial x_j} \qquad\qquad \forall u \in C^\infty, \quad \forall v \in C_0^\infty(\mathbf{R}^N)$$

Derivata debole. Sia $u \in L^1_{loc}(\mathbf{R}^N)$ (cioé $\int_{B_R} |u| < +\infty \quad \forall R > 0$). Se

$$\exists \quad u_j \in L^1_{loc}(\mathbf{R}^N) : \qquad \int_{\mathbf{R}^N} u_j\; v \;=\; -\int_{\mathbf{R}^N} u\; \frac{\partial v}{\partial x_j} \qquad\qquad \forall v \in C_0^\infty(\mathbf{R}^N)$$

diremo che u_j é la derivata debole di u rispetto alla j-esima variabile e scriveremo $\frac{\partial u}{\partial x_j} = u_j$. Se u ha derivate deboli $\frac{\partial u}{\partial x_j} = u_j$, $j = 1, \ldots, N$, scriveremo $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N})$.

NOTA. É una **buona definizione:**

(i) $u_j, v_j \in L^1_{loc}(\mathbf{R}^N)$, $\int_{\mathbf{R}^N} u_j\, v = -\int_{\mathbf{R}^N} u\, \frac{\partial v}{\partial x_j} = \int_{\mathbf{R}^N} v_j\, v \quad \forall v \in C_0^\infty(\mathbf{R}^N) \;\Rightarrow$ $\int_{\mathbf{R}^N} (u_j - v_j)\, v = 0 \quad \forall v \in C_0^\infty(\mathbf{R}^N) \;\Rightarrow\; (u_j - v_j) = 0$.
In particolare, ogni funzione C_0^∞ ha derivate deboli che coincidono con le derivate usuali.

(ii) $u_n \to u$ in L^1_{loc}, $\partial_j u_n \to v_j$ in L^1_{loc} $\quad\Rightarrow\quad$ u ha derivate deboli $\partial_j u = v_j$
Infatti

$$\int u\;\; \partial_j \varphi \;=\; \lim_n \int u_n\;\; \partial_j \varphi \;=\; -\lim_n \int \varphi\;\; \partial_j u_n \;=\; -\int \varphi\;\; v_j$$

Definizione.

$H^1(\mathbf{R}^N)$ é lo spazio delle funzioni in L^2 con derivate deboli in L^2, dotato del prodotto scalare e relativa norma

$$< u, v >_{H^1} = \int_{\mathbf{R}^N} u\, v + < \nabla u, \nabla v >, \qquad\qquad \|u\|^2_{H^1} \;=\; \int_{\mathbf{R}^N} |u|^2 \;+\; |\nabla u|^2$$

$\mathcal{D}^1(\mathbf{R}^N)$ é lo spazio delle funzioni in $L^{\frac{2N}{N-2}}$ con derivate deboli in L^2, dotato del prodotto scalare e relativa norma

$$< u, v >_{\mathcal{D}^1} = \int_{\mathbf{R}^N} < \nabla u, \nabla v > \qquad\qquad \|u\|^2_{\mathcal{D}^1} \;=\; \int_{\mathbf{R}^N} |\nabla u|^2$$

H^1 **é un Hilbert**

Sia u_n di Cauchy in H^1, ovvero $u_n,\ \partial_j u_n$ sono di Cauchy in L^2. Dunque esistono $u \in L^2,\ u_j \in L^2$ tali che

$$u_n \to u \quad \text{in} \quad L^2, \quad \partial_j u_n \to u_j \quad \text{in} \quad L^2$$

Allora, (vedi Nota-ii), u ha derivate deboli in L^2 date da $\partial_j u = u_j$, ovvero $u \in H^1$.

Prop. 1.

$$(i) \quad \varphi \in C_0^\infty,\ u \in H^1 \ \Rightarrow\ \varphi * u \in H^1 \quad \text{e} \quad \frac{\partial}{\partial x_j}(\varphi * u) \ =\ \varphi * \frac{\partial u}{\partial x_j}$$

$$(ii) \quad \varphi \in C_0^\infty,\ u \in \mathcal{D}^1 \ \Rightarrow\ \varphi * u \in \mathcal{D}^1 \quad \text{e} \quad \frac{\partial}{\partial x_j}(\varphi * u) \ =\ \varphi * \frac{\partial u}{\partial x_j}$$

Infatti, $\quad u,\ \frac{\partial u}{\partial x_j} \in L^2 \ \Rightarrow\ \varphi * u, \quad \varphi * \frac{\partial u}{\partial x_j} \in C^\infty \cap L^2 \quad$ e

$$\int_{\mathbf{R}^N} \psi\, \partial_j(\varphi * u) \ =\ \int_{\mathbf{R}^N} \psi\, \partial_j\varphi * u \ =\ \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} (\partial_j\varphi)(x-y)u(y)\, dy \right) \psi(x)\, dx \ =$$

$$= \ -\int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} \frac{\partial}{\partial y_j}\varphi(x-y)u(y)\, dy \right) \psi(x)\, dx \ =\ \int_{\mathbf{R}^N} (\varphi * \partial_j u)\, \psi \quad \forall \psi \in C_0^\infty(\mathbf{R}^N)$$

Prop. 2.

$$(i) \quad \forall u \in H^1,\ \exists u_n \in C_0^\infty(\mathbf{R}^N): \quad u_n \to u, \quad \partial_j u_n \to \partial_j u \quad \text{in} \quad L^2$$

$$(ii) \quad \forall u \in \mathcal{D}^1,\ \exists u_n \in C_0^\infty(\mathbf{R}^N): \quad u_n \to u \quad \text{in} \quad L^{\frac{2N}{N-2}}, \quad \partial_j u_n \to \partial_j u \quad \text{in} \quad L^2$$

Prova. (i) Sia $\varphi \in C_0^\infty$ nucleo regolarizzante: $\varphi_\epsilon * u \in C_0^\infty$, $\varphi_\epsilon * u \to u$, $\partial_j(\varphi_\epsilon * u) = \varphi_\epsilon * \partial_j u \to \partial_j u$ in L^2. Le $\varphi_\epsilon * u$ non saranno peró in generale a supporto compatto. Basta peró osservare che

$$\forall v \in C^\infty \cap L^2,\ \exists v_n \in C_0^\infty: \quad v_n \to v, \quad \partial_j v_n \to \partial_j v \quad \text{in} \quad L^2$$

Sia infatti

$\psi \in C_0^\infty, 0 \le \psi \le 1, \psi = 1$ if $|x| \le 1, \psi = 0$ if $|x| \ge 2, \psi_\epsilon(x) = \psi(\epsilon x) \to 1 \forall x \in \mathbf{R}^N$

Allora (prenderemo $v_n(x) = v(x)\psi(\frac{x}{n})$)

$$v(x)\psi(\epsilon x) \to u(x)\ \forall x \in \mathbf{R}^N, |v\psi_\epsilon| \le |v| \in L^2 \ \Rightarrow\ v\psi_\epsilon \to_{\epsilon\to 0} v \quad \text{in} \quad L^2$$

$$\partial_j(v\psi_\epsilon) = \psi_\epsilon \partial_j v + v\partial_j\psi_\epsilon \to_{\epsilon\to 0} \partial_j v \quad \text{in} \quad L^2$$

perché $\psi_\epsilon\partial_j v \to_{\epsilon\to 0} \partial_j v$ puntualmente e $|\psi_\epsilon\, \partial_j v| \le |\partial_j v|$ implica $\psi_\epsilon\partial_j v \to_{\epsilon\to 0} \partial_j v$ in L^2 (convergenza dominata), mentre

$$\int |v\, \partial_j\psi_\epsilon|^2 = \epsilon^2 \int v^2\, |\partial_j\psi|^2(\epsilon x) \le \epsilon^2\, \|\partial_j\psi\|_\infty^2 \int v^2 \to_{\epsilon\to 0} 0$$

(ii) Qui, $\varphi_\epsilon * v \to v$ in $L^{\frac{2N}{N-2}}$, $\partial_j(\varphi_\epsilon * v) = \varphi_\epsilon * \partial_j v \to \partial_j v$ in L^2.
Come in (i) basta mostrare che

$$\forall v \in C^\infty \cap L^{\frac{2N}{N-2}},\ \exists v_n \in C_0^\infty : \quad v_n \to v \text{ in } L^{\frac{2N}{N-2}}, \quad \partial_j v_n \to \partial_j v \quad \text{in} \quad L^2$$

Come in (i), $v\psi_\epsilon \to_{\epsilon\to 0} v$ in $L^{\frac{2N}{N-2}}$ e $\psi_\epsilon\partial_j v \to_{\epsilon\to 0} \partial_j v$ in L^2, mentre

$$\int |v\,(\partial_j\psi_\epsilon)|^2 \le \int\limits_{1\le|\epsilon x|\le 2} u^2\,\epsilon^2\,|\partial_j\psi|^2(\epsilon x) \le \epsilon^2 \left(\int\limits_{1\le|\epsilon x|\le 2} |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \left(\int |\partial_j\psi|^N(\epsilon x)\right)^{\frac{2}{N}}$$

$$\le \left(\int\limits_{1\le|\epsilon x|\le 2} |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} (\left(\int |\partial_j\psi|^N\right)^{\frac{2}{N}} \to_{\epsilon\to 0} 0$$

NOTA $H^1 \subset \mathcal{D}^1$, inclusione stretta. Infatti, se $u \in H^1$, $u_n \in C_0^\infty$, $u_n \to u$ in L^2, $\partial_j u_n \to \partial_j u$ in L^2, da Sobolev $\left(\int |u_n|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \le \int |\nabla u_n|^2$ e quindi, per Fatou, $u \in L^{\frac{2N}{N-2}}$.

Poi, se $U(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}$, $N \ge 3$, é $U \in L^{\frac{2N}{N-2}}$, $|\nabla U|^2 = (N-2)^2 \frac{|x|^2}{(1+|x|^2)^N}$ e quindi $|\nabla U| \in L^2$, mentre $\int U^2 = \int \frac{1}{(1+|x|^2)^{N-2}} < +\infty \Leftrightarrow N > 4$

Corollario

$$u,\ v \in H^1 \quad \Rightarrow \quad \int\limits_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\ v = -\int\limits_{\mathbf{R}^N} u\ \frac{\partial v}{\partial x_j}$$

L'uguaglianza vale per approssimanti u_n, v_n e, nelle ipotesi fatte, é lecito passare al limite.

Diseguaglianza di Sobolev $\left(\int\limits_{\mathbf{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \le c(N) \int\limits_{\mathbf{R}^N} |\nabla u|^2 \quad \forall u \in \mathcal{D}^1$

Infatti, $u_n \in C_0^\infty, u_n \to u$ in $\mathcal{D}^1$,

$$\left(\int\limits_{\mathbf{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \le \liminf_n \left(\int\limits_{\mathbf{R}^N} |u_n|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \le \liminf_n c(N) \int\limits_{\mathbf{R}^N} |\nabla u_n|^2 = c(N) \int\limits_{\mathbf{R}^N} |\nabla u|^2$$

$\mathcal{D}^1$ **é Hilbert**. Se infatti u_n é di Cauchy in $\mathcal{D}^1$, ovvero $\partial_j u_n$ sono di Cauchy in L^2, per la diseguaglianza di Sobolev u_n é di Cauchy in $L^{\frac{2N}{N-2}}$. Dunque esistono $u \in L^{\frac{2N}{N-2}}$, $u_j \in L^2$ tali che

$$u_n \to u \quad \text{in} \quad L^{\frac{2N}{N-2}}, \quad \partial_j u_n \to u_j \quad \text{in} \quad L^2$$

Allora, vedi Nota-ii, u ha derivate deboli $\partial_j u = u_j$ e quindi $u \in \mathcal{D}^1$ e .

Esempio. Sia $h \in L^{\frac{2N}{N+2}}$. Allora

$$h * G_{N-2} \in \mathcal{D}^1 \quad \text{e} \quad \partial_j \int \frac{h(y)dy}{|x-y|^{N-2}} = -(N-2) \int \frac{h(y)(x_j - y_j)dy}{|x-y|^N}$$

In primo luogo, da (HLS) segue: $\|h * G_{N-2}\|_{\frac{2N}{N-2}} \le d(N)\|h\|_{\frac{2N}{N+2}}$.

Poi, se $h \in C_0^\infty(B_R)$,

$$\left|\frac{\partial h}{\partial x_j}(x-z)\frac{1}{|z|^{N-2}}\right| \le \left\|\frac{\partial h}{\partial x_j}\right\|_\infty \frac{1}{|z|^{N-2}}\chi_{B_R} \Rightarrow \frac{\partial}{\partial x_j}(h * G_{N-2}) = \int \frac{\partial h}{\partial x_j}(x-z)\frac{dz}{|z|^{N-2}}$$

$$= \lim_{\epsilon \to 0} \int \frac{\partial h}{\partial x_j}(x-z)\frac{dz}{(\epsilon^2 + |z|^2)^{\frac{N-2}{2}}} = \lim_{\epsilon \to 0} \int \frac{\partial h}{\partial x_j}(y)\frac{dy}{(\epsilon^2 + |x-y|^2)^{\frac{N-2}{2}}} =$$

$$-\lim_{\epsilon \to 0} \int h(y)\frac{-(N-2)(x_j - y_j)}{(\epsilon^2 + |x-y|^2)^{\frac{N}{2}}}\, dy = (N-2)\int h(y)\frac{x_j - y_j}{|x-y|^N}\, dy$$

perché $\left|\frac{h(x-z)}{|z|^{N-1}}\right| \le \|h\|_\infty \frac{1}{|z|^{N-1}}\chi_{B_R}$. In particolare, ancora (HLS), dá

$$\left\|\frac{\partial}{\partial x_j}(h * G_{N-2})\right\|_2 = (N-2)\|h * G_{N-1}\|_2 \le d(N)\|h\|_{\frac{2N}{N+2}}$$

Allora $h_n \in C_0^\infty, h_n \to h$ in $L^{\frac{2N}{N+2}} \Rightarrow \|\frac{\partial}{\partial x_j}(h_n * G_{N-2}) - \frac{\partial}{\partial x_j}(h_m * G_{N-2})\|_2 \le a(N)\|h_n - h_m\|_{\frac{2N}{N-2}} \Rightarrow h_n * G_{N-2}$ é di Cauchy in $L^2 \Rightarrow h * G_{N-2} \in \mathcal{D}^1$ e

$$\frac{\partial}{\partial x_j}(h * G_{N-2}) = \lim_n \frac{\partial}{\partial x_j}(h_n * G_{N-2}) = \lim_n (N-2)\int h_n(y)\frac{x_j - y_j}{|x-y|^N}\, dy$$

$$= (N-2)\int h(y)\frac{x_j - y_j}{|x-y|^N}\, dy$$

Problemi e complementi XI Settimana

Problema 1. Sia

$$\lambda(R) := \inf\{\int_{\mathbf{R}^N} |\nabla u|^2 : \quad u \in C_0^\infty(B_R), \quad \int_{\mathbf{R}^N} |u|^2 = 1\}$$

Esprimere, utilizzando cambi di scala, $\lambda(R)$ in funzione di R e di $\lambda(1)$ e descrivere il comportamento di $\lambda(R)$ al tendere di R a zero e a $+\infty$

Problema 2. Sia $N \geq 3$. Sia

$$S := \inf\{\int_{\mathbf{R}^N} |\nabla u|^2 : \quad u \in C_0^\infty(\mathbf{R}^N), \quad \int_{\mathbf{R}^N} |u|^{\frac{2N}{N-2}} = 1\}$$

$$S(R) := \inf\{\int_{\mathbf{R}^N} |\nabla u|^2 : \quad u \in C_0^\infty(B_R), \quad \int_{\mathbf{R}^N} |u|^{\frac{2N}{N-2}} = 1\}$$

Provare che $S(R)$ non dipende da R.

Dedurre che esistono successioni $u_n \in C_0^\infty(B_1)$ tali che $\sup_n \int_{\mathbf{R}^N} |\nabla u_n|^2 < +\infty$ e che non hanno estratte convergenti in $L^{2N}N-2(\mathbf{R}^N)$

Esercizio 1. Esistono successioni $u_n \in C_0^\infty(\mathbf{R}^N)$ tali che $\sup_n \int_{\mathbf{R}^N} |\nabla u_n|^2 < +\infty$ e che non hanno estratte convergenti in $L^2(\mathbf{R}^N)$?

Suggerimento. Considerare $f(x) = \frac{1}{\sqrt{|x|}}\chi_{[-1,1]}$