AM2: Tracce delle lezioni- VII Settimana

SERIE DI POTENZE NEL CAMPO COMPLESSO

1. Definizione $z_n \in \mathbf{C}$ converge a z $(z_n \to_n z)$ $\Leftrightarrow$ $|z_n - z| \to_n 0$.

Siccome $|z_n - z|^2 = |Rez_n - Rez|^2 + |Imz_n - Imz|^2$, si ha che:

$$z_n \to_n z \quad \Leftrightarrow \quad Rez_n \to_n Rez \quad \text{e} \quad Imz_n \to_n Imz.$$

Per la stessa ragione, valgono, ad esempio, le seguenti proprietá:

$z_n \to_n z \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists n_\epsilon : |z_n - z_m| \le \epsilon$ se $n, m \ge n_\epsilon$ (Cauchy)
$z_n \to_n z, \quad w_n \to_n w \quad \Rightarrow \quad z_n + w_n \to z + w, \quad z_n\, w_n \to z\, w$
$z_n \to_n z \quad \Rightarrow \quad \sup_n |z_n| < +\infty$ (z_n é limitata)
$\sup_n |z_n| < +\infty \quad \Rightarrow \quad z_n$ ha una sottosuccessione convergente.
$z_n \to_n z \quad \Rightarrow \quad \overline{z_n} \to_n \overline{z} \quad (\overline{a + ib} := a - ib)$.

2. Definizione $\sum_{n=1}^{\infty} z_n$ converge sse $S_N := \sum_{n=1}^{N} z_n$ converge.
$\sum_n z_n$ si dice assolutamente convergente se $\sum_n |z_n| < +\infty$.

(Cauchy) $\sum_n z_n$ converge $\Leftrightarrow$ $\forall \epsilon > 0,\ \exists N_\epsilon :\ |\sum_{n=N}^{N+p} z_n| \le \epsilon \quad \forall N \ge N_\epsilon,\ \forall\, p$.
In particolare, $\sum_n |z_n| < +\infty \quad \Rightarrow \quad \sum_n z_n$ converge. In particolare $\limsup_n |z_n|^{\frac{1}{n}} < 1 \quad \Rightarrow \quad \sum z_n$ converge. Come nel caso reale si ha quindi

3. Formula di Cauchy-Hadamard Sia $a_n \in \mathbf{C}$, $r := \limsup_n |a_n|^{-\frac{1}{n}}$:

$$z \in \mathbf{C}, \quad |z| < r \ \Rightarrow \ \sum_{n=0}^{\infty} |a_n z^n| \ < \ +\infty \quad , |z| > r \ \Rightarrow \ \sum_{n=0}^{\infty} |a_n z^n| \ = \ +\infty$$

r:= raggio di convergenza, $D_r := \{z :\ |z| < r\}$:= disco di convergenza .

ESEMPI. $\sum_{n=0}^{\infty} z^n$ converge in $|z| < 1$ e $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.
$\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converge per ogni z.

4. Definizione $\exp\, z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad z \in \mathbf{C}$

Questa sará assunta d'ora in avanti come definizione della funzione esponenziale (in ambito reale o complesso). In particolare, $\sum_{n=0}^{\infty} \frac{1}{n!}$ sará la definizione del numero di Nepero. Nel seguito deriveremo le proprietá dell'esponenziale (reale o complesso).

Sia $\quad f(z) := \sum_{n=0}^{\infty} a_n z^n \quad z \in D_r \quad$ Tale posizione definisce una funzione di variabile complessa a valori complessi.

5. Continuitá. $f : D_r \to \mathbf{C}$ é continua in $z_0 \in D_r$ se:

$$\forall \epsilon > 0 \, \exists \delta > 0 : \quad |z - z_0| \le \delta \Rightarrow |f(z) - f(z_0)| \le \epsilon$$

ovvero se $\quad z_n \to_n z \quad \Rightarrow \quad f(z_n) \to_n f(z)$

6. Derivabilitá. $f : D_r \to \mathbf{C}$ é derivabile in $z_0 \in D_r$ con derivata $f'(z_0)$ se

$$\forall \epsilon > 0 \, \exists \delta > 0 : \quad |z - z_0| \le \delta \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \le \epsilon$$

Anche qui, come nel caso reale: f é derivabile in $z_0 \Rightarrow f$ é continua in z_0.

7. Esempio $\quad f(z) = z^n \quad$ é continua e derivabile, con $\quad \frac{dz^n}{dz} = nz^{n-1}$

Prova. Se $|z_0| = \frac{tr}{1+\delta}$, $t < 1$, $\delta > 0$, $|z - z_0| \le \delta |z_0|$ e quindi $|z| \le tr$, allora

$$(*) \quad |z^n - z_0^n| \le n|z - z_0| \, (tr)^{n-1} \qquad (**) \quad \left|\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1}\right| \le (n-1)^2 |z - z_0| (tr)^{n-2}$$

É $\quad |z^n - z_0^n| = |(z - z_0)(z^{n-1} + z^{n-2} z_0 + \ldots + z \, z_0^{n-2} + z_0^{n-1})| \le |z - z_0| \, n \, (tr)^{n-1}$,

$\left|\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1}\right| = |z^{n-1} - z_0^{n-1} + z_0(z^{n-2} - z_0^{n-2}) + \ldots + z_0^{n-2}(z - z_0)| \le$

$\le |z^{n-1} - z_0^{n-1}| + |z_0||z^{n-2} - z_0^{n-2}| + \ldots + |z_0|^{n-2}|z - z_0| \le$

$\le |z - z_0| [(n-1)(tr)^{n-2} + (n-2)|z_0|(tr)^{n-3} + \ldots + |z_0|^{n-2}] \le (n-1)^2 |z - z_0| (tr)^{n-2}$.

8. Proposizione. $\quad f(z) := \sum_{n=0}^{\infty} a_n z^n \quad z \in D_r \quad$ é $\quad C^{\infty}(D_r)$.

Proviamo che $\quad \frac{d^k f}{dz^k}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} a_{j+k} z^j \quad \forall z \in D_r$. $\quad$ Come nel caso reale, basta provare la formula per $k = 1$. Da $(**)$, si ottiene

$\left|\frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1}\right| \le \sum_{n=2}^{\infty} |a_n| \left|\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1}\right| \le$

$|z - z_0| \sum_{n=2}^{\infty} |a_n| \, (n-1)^2 \, (tr)^{n-2} \to_{z \to z_0} 0$, perché $\sum_{n=2}^{\infty} |a_n| \, (n-1)^2 \, (tr)^{n-2} < +\infty$.

9. $\quad \exp(z + w) = \exp z + \exp w \quad \forall z, w \in \mathbf{C}$. $\quad$ Seguirá da

10. Lemma (Prodotto secondo Cauchy). $\sum_{n=0}^{\infty} |z_n| + \sum_{n=0}^{\infty} |w_n| < +\infty \Rightarrow$

$$\sum_{n=0}^{\infty} \Big| \sum_{j+k=n} z_j\, w_k \Big| < +\infty \ , \qquad \sum_{n=0}^{\infty} \Big(\sum_{j+k=n} z_j\, w_k \Big) = \Big(\sum_{n=0}^{\infty} z_n \Big) \Big(\sum_{n=0}^{\infty} w_n \Big)$$

Siano $s_N := \sum_{n=0}^{N} z_n$, $\sigma_N := \sum_{n=0}^{N} w_n$ $\quad p_n := \sum_{n=0}^{N} \big(\sum_{j+k=n} z_j\, w_k \big) =$

$= z_0\, w_0 + (z_0\, w_1 + z_1\, w_0) + \ldots + (z_0\, w_N + z_1\, w_{N-1} + \ldots + z_{N-1}\, w_1 + z_N\, w_0) =$

$z_0\,(w_0 + w_1 + \ldots + w_N) + z_1\,(w_0 + \ldots + w_{N-1}) + \ldots + z_N\, w_0.$ Dunque

$$|s_N\, \sigma_N - p_N| =$$

$|z_0(w_0+\ldots+w_N)+z_1(w_0+\ldots+w_N)+\ldots+z_{N-1}(w_0+\ldots+w_N)+z_N(w_0+\ldots+w_N)-$

$[z_0\,(w_0 + w_1 + \ldots + w_N) + z_1\,(w_0 + \ldots + w_{N-1}) + \ldots + z_{N-1}(w_0 + w_1) + z_N\, w_0]| =$

$|z_1\, w_N + z_2(w_{N-1} + w_N) + \ldots + z_{N-1}(w_2 + \ldots + w_N) + z_N\,(w_1 + \ldots + w_N)| \leq$

$$\leq \sum_{j=1}^{n} |z_j|\, |w_{N-j+1} + \ldots + w_N| + \sum_{j=n+1}^{N} |z_j|\, |w_{N-j+1} + \ldots + w_N| \leq$$

$\leq \sup_{j \leq n} |z_j| \big(\sum_{k=N-n+1}^{\infty} |w_k| \big) + \sup_{j \geq n+1} |z_j| \big(\sum_{k=1}^{\infty} |w_k| \big) \quad n := [\frac{N}{2}].$ Da

$\sum_{k=N-[\frac{N}{2}]+1}^{\infty} |w_k| \to_{N\to+\infty} 0,\ \sup_{j \geq [\frac{N}{2}]+1} |z_j| \to_{N\to+\infty} 0,\ \sup_j |z_j| < +\infty,\ \sum_{k=1}^{\infty} |w_k| < \infty$

segue $|s_N\, \sigma_N - p_N| \to_{N\to+\infty} 0$ e quindi $\lim_N p_N = \lim_N s_N\, \sigma_N$.

Prova di 9. $\exp\,(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} =$

$$\sum_{n=0}^{\infty} \Big(\frac{1}{n!} \sum_{j+k=n} \frac{n!}{j!\, k!} z^j\, w^k \Big) = \sum_{n=0}^{\infty} \Big(\sum_{j+k=n} \frac{z^j}{j!} \frac{w^k}{k!} \Big) = \Big(\sum_{n=0}^{\infty} \frac{z^j}{j!} \Big) \Big(\sum_{n=0}^{\infty} \frac{w^k}{k!} \Big) = \exp z\, \exp w$$

ESERCIZIO. $e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to+\infty} (1 + \frac{1}{n})^n$ É

$(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} < \sum_{k=0}^{n} \frac{1}{k!}$ perché $\frac{n!}{n^k\,(n-k)!} = \frac{(n-k)!(n-k-1)\ldots n}{(n-k)!n\, n \ldots n} < 1$

e quindi $\limsup_n (1 + \frac{1}{n})^n \leq \sum_{n=0}^{\infty} \frac{1}{n!}$; Viceversa, $n > n_0 \Rightarrow$

$(1+\frac{1}{n})^n > \sum_{k=0}^{n_0} \frac{n!}{k!\,(n-k)!}\frac{1}{n^k} \;\;\Rightarrow\;\; \liminf_n (1+\frac{1}{n})^n \geq \sum_{k=0}^{n_0} \frac{1}{k!} \lim_n \frac{n!}{n^k\,(n-k)!} \;=\; \sum_{k=0}^{n_0} \frac{1}{k!}, \;\; \forall n_0$
perché $\frac{n!}{n^k\,(n-k)!} = (1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n}) \to_n 0$ e quindi $\liminf_n (1+\frac{1}{n})^n \geq \sum_{k=0}^{\infty} \frac{1}{k!}$

11. (i) $\exp(-z) \;=\; (\exp z)^{-1}$ $\qquad(ii)$ $\exp r \;=\; e^r \;\;\forall r \in \mathbf{Q}$.
(iii) $\overline{\exp z} \;=\; \exp\,\overline{z}$, $\qquad(iv)$ $|\exp\,(ix)| \;=\; 1 \;\;\forall x \in \mathbf{R}$.

In particolare, $x \to \exp x,\;\; x \in \mathbf{R}$ é prolungamento continuo di $r \to e^r, r \in \mathbf{Q}$.

(i) $1 \;=\; \exp\,(z-z) \;=\; \exp z\;\exp\,(-z)$ $\qquad(ii)$ se $p \in \mathbf{N}, z \in \mathbf{C}$, $(\exp z)^p = \exp\,(pz) \;\;\forall p \in \mathbf{N}$ e in particolare $(\exp\,(\frac{1}{p}))^p = e$ ovvero $\exp\,(\frac{1}{p}) \;=\; e^{\frac{1}{p}}$. Quindi $\exp\,(\frac{p}{q}) = (\exp\frac{1}{q})^p = (e^{\frac{1}{q}})^p = e^{\frac{p}{q}}$.

(iii) $\exp\,\overline{z} \;=\; \lim_{N\to+\infty} \overline{\sum_{n=0}^{N} \frac{z^n}{n!}} \;=\; \lim_{N\to+\infty} \sum_{n=0}^{N} \overline{\frac{z^n}{n!}} \;=\; \overline{\lim_{N\to+\infty} \sum_{n=0}^{N} \frac{z^n}{n!}} \;=\; \overline{\exp z}$.

(iv) da (i) segue $|\exp\,(ix)| \;=\; |\exp\,(-ix)|$. Ma $|\exp\,(-ix)| \;=\; |\exp\,(ix)|^{-1}$.

Per quanto visto, $\exp\,(x+iy) = \exp x\;\exp\,(iy) \;=\; (\sum_0^{\infty} \frac{x^n}{n!})\;\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}$ $\quad x, y \in \mathbf{R}$. É

$$Re\,(\exp\;\; iy) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} \qquad\qquad Im\,(\exp\;\; iy) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$$

12. Definizione $\qquad \cos y := Re(\exp\; iy) \qquad \sin y := Im(\exp\; iy) \qquad y \in \mathbf{R}$

Notiamo che $\frac{d}{dx} Im(\exp iy) = Re(\exp iy) \qquad \frac{d}{dx} Re(\exp iy) = -Im(\exp iy)$

13. $\exists T > 0:$ $\exp\,(iT) \;=\; 1$, $\exp\,(it) \;\neq\; 1 \;\;\forall t \in (0,T)$, $\exp\,[i(t+T)] \;=\; \exp\,(it) \;\;\forall t$ $\left(\pi \;:=\; \frac{T}{2}\right)$. Inoltre, $\exp(i\theta) = 1 \;\;\Leftrightarrow\;\; \frac{\theta}{2\pi} \in \mathbf{Z}$.

Infatti, $\cos 0 = 1 \;\;\Rightarrow\;\; \exists \underline{t}:\; \cos t \geq \frac{1}{2} \;\;\forall t \in [0, \underline{t}] \;\;\Rightarrow\;\; \sin \underline{t} = \int_0^{\underline{t}} \cos\tau\, d\tau \;\geq\; \frac{1}{2}\underline{t}$.
$\cos t > 0\;\forall t \in [\underline{t}, \overline{t}] \;\Rightarrow\; \sin \underline{t} \;\geq\; \frac{1}{2}\underline{t}\;\forall t \in [\underline{t}, \overline{t}] \;\Rightarrow\; -2 \leq \cos\overline{t} - \cos\underline{t} \;=\; -\int_{\underline{t}}^{\overline{t}} \sin t\, dt \;\leq\; -(\overline{t} - \underline{t})\,\sin\underline{t} \;\leq\; -\frac{1}{2}\underline{t}(\overline{t} - \underline{t}) \;\;\Rightarrow\;\; \overline{t} \;\leq\; \underline{t} + \frac{4}{\underline{t}} \;\;\Rightarrow\;\; \exists t \leq \underline{t} + \frac{4}{\underline{t}} < +\infty:\; \cos t = 0$.
Sia $\frac{T}{4} := \inf\{t:\;\; \cos t = 0\}$. Da $\cos\frac{T}{4} = 0$ segue $\exp(i\frac{T}{4}) = i$ e quindi $\exp(iT) = 1$.
Inoltre, $t \in (0,T) \Rightarrow \exp(it) \neq 1$ perché $\exp\,(it) = 1 \Rightarrow \exp(i\frac{t}{4}) = \pm 1$ op. $\exp(i\frac{t}{4}) = \pm i \Rightarrow \cos\frac{t}{4} = \pm 1$ oppure $\cos\frac{t}{4} = 0$. Ma $0 < t < T \Rightarrow \cos\frac{t}{4} \in (0,1)$.
Poi, $\exp\,[i(t+T)] \;=\; \exp\,(it)\;\exp(iT) \;=\; \exp\,(it) \;\;\forall t$. Infine,
$\theta \in (2k\pi, 2(k+1)\pi) \;\;\Rightarrow \exp(i\theta) = \exp[i(\theta - 2k\pi)] \neq 1$ perché $\theta - 2k\pi \in (0, 2\pi)$.

NOTA. É $\exp(z + 2k\pi i) \;=\; \exp z\;\exp(2k\pi i) \;=\; \exp z$: la funzione $\exp z$ é periodica di periodo (complesso) $2\pi i$.

14. (i) $\forall z \in \mathbf{C}$, con $|z| = 1$, $\exists\,!\, t \in (-\pi, \pi]$: $z = \exp(it)$. Posto $arg\, z := t$, $Arg\, z := \{arg\, z + 2k\pi,\ k \in \mathbf{Z}\}$, é $z = \exp(i\, arg\, z) = \exp(i\, Arg\, z)$.

(ii) Sia $w = |w| \exp(i\, arg\, w)$. Allora
$$w = \exp z \ \Leftrightarrow\ z \in \{\log|w| + i\, Arg\, w\}.$$

(i) $\cos t$ é biiezione tra $[0, \pi]$ e $[-1, 1]$. Infatti $\sin t = 0$, $\sin s > 0$ in $[0, t] \Rightarrow \exp(it) = -1 \Rightarrow \exp(\frac{t}{2}i) = \sqrt{-1} \Rightarrow \cos\frac{t}{2} = 0 \Rightarrow t \geq \pi$ e cioé $(\cos t)' = -\sin t < 0$ in $(0, \pi)$. Inoltre $\exp(i\pi) = \exp(\frac{\pi}{2}i)^2 = -1 \Rightarrow \cos\pi = -1$. Infine, $\cos 0 = 1$.

Allora, $z = x + iy$, $x^2 + y^2 = 1$, $\Rightarrow |x| \leq 1 \Rightarrow \exists\,!\, t \in [0, \pi] :\ x = \cos t$, $y = \sqrt{1 - \cos^2 t} = \sin t$, se $y \geq 0$. Se $y < 0$, $\overline{z} = \exp(it)$, $t \in (0, \pi)$ e quindi $z = \exp(i\tau), \tau = -t \in (-\pi, 0)$.

(ii) $\exp z = w \Leftrightarrow \exp(Re\, z)\exp(i\, Im\, z) = |w| \exp(i \arg w) \Leftrightarrow \exp Re\, z = \log|w|$ e $\frac{Im\, z - t}{2\pi} \in \mathbf{Z} \Leftrightarrow z \in \{\log|w| + i\, Arg\, w\}$.

15. Definizione di logaritmo in C $\forall w \neq 0$, $Log\, w := \{\log|w| + i\, Arg\, w\}$

La funzione $log\, w := log\,|w| + i\, arg\, w$ si chiama valore principale del logaritmo.

Esempi. $Log\, x = \log x + 2k\pi i$, $\forall x > 0$, $Log\, x = \log|x| + (2k+1)\pi i$, $\forall x > 0$. $Log\,(-1) = \pi i$, $Log\, i = \frac{\pi}{2} i$, $Log\,(1 - i) = \log\sqrt{2} + (2k - \frac{1}{4})i$.

Esercizio. $Log\,(zw) = Log\, z + Log\, w$ ove, dati $A, B \subset \mathbf{C}$, $A + B := \{a + b :\ a \in A,\ b \in B\}$. Infatti $Arg\,(zw) = Arg\, z + Arg\, w$.
$Log\,(-z) = \{\log z + (2k+1)\pi i\}\ \forall z \neq 0$.
Trovare l'errore in $z^2 = (-z)^2 \Rightarrow Log\,(z)^2 = Log\,(-z)^2 \Rightarrow Log z + Log z = Log(-z) + Log(-z) \Rightarrow 2 Log z = 2 Log(-z) \Rightarrow Log z = Log(-z)$.

15. Potenze in C Se $a, z \in C$, $a \neq 0$
$a^z := \exp(z\, Log\, a) = \{\exp[z(\log|a| + i(arg\, a + 2k\pi)\}$

Esempi. Se $z = n \in \mathbf{N}$, $a^p = a \times \ldots \times a$ (p volte).
Se $z = \frac{1}{n}$, $n \in \mathbf{N}$, $a^{\frac{1}{n}} = \{|a|^{\frac{1}{n}} \exp\frac{arg\, a + 2k\pi}{n},\ k = 0, \ldots, n-1\}$ (le n radici complesse di a).
Se $z \notin \mathbf{Q}$, a^z é un insieme infinito. In particolare, $e^z = \exp z$ se e solo se $z = 0, 1, \ldots$.

16. Definizione

$$\cos z \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z \;=\; \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbf{C}$$

$$\cosh z \;=\; \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \sinh z \;=\; \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbf{C}$$

Dunque $\cosh z \,+\, \sinh z \;=\; \exp z$, $\exp(-z) \;=\; \cosh z \,-\, \sinh z$, da cui

$$\cosh z \;=\; \frac{1}{2}(\exp z \;+\exp(-z)), \qquad \sinh z \;=\; \frac{1}{2}(\exp z \;-\exp(-z))$$

16. Le formule di Eulero

(i) $e^{iz} \;=\; \cos z \,+\, i \sin z, \quad e^{-iz} \;=\; \cos z \,-\, i \sin z \quad \forall\, z \in \mathbf{C}$

(ii) $\cos z \;=\; \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z \;=\; \frac{e^{iz}-e^{-iz}}{2i} \quad \forall\, z \in \mathbf{C}$

(i) Infatti exp $(iz) \;=\; \sum_{n=0}^{\infty}(i)^{2n}\frac{z^{2n}}{2n!} \;+\; \sum_{n=0}^{\infty}(i)^{2n+1}\frac{z^{2n+1}}{(2n+1)!} \;=\; (\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2n!}) \;+\; i \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} = \cos z \,+\, i\, \sin z$. In particolare,

$$e^{it} \;=\; \cos t \,+\, i \sin t \quad \forall\, t \in \mathbf{R}, \qquad e^{-it} \;=\; \cos t \,-\, i \sin t \quad \forall\, t \in \mathbf{R}$$

Sommando (sottraendo) le formule in (i), si ottiene (ii).

Usando (ii), troviamo

$$\cos z \;=\; \cosh iz, \qquad \sinh iz \;=\; i \sin z$$

ESERCIZIO.

Provare che $\sin z, \cos z$ sono funzioni 2π periodiche, e che $\sin^2 z \,+\, \cos^2 z \equiv 1$.

Provare che $\sinh z, \cosh z$ sono $2\pi i$ periodiche e $\cosh^2 z \,-\, \sinh^2 z \equiv 1$