

SOLUZIONI DEGLI ESERCIZI SUI LIMITI DI FUNZIONI

ESERCIZIO 1

(a) $\lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} e^{\ln x^x} = \lim_{x \rightarrow 0} e^{x \ln x} = 1$

(b) Poniamo $x = \frac{1}{n}$, quindi

$$\lim_{x \rightarrow 0} x \ln x = \lim_{n \rightarrow \infty} \frac{\ln(1) - \ln n}{n} = \lim_{n \rightarrow \infty} -\frac{\ln n}{n} = 0.$$

(c) Poniamo $x = \frac{1}{n}$, quindi

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n \ln\left(1 + \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right)^n = 1.$$

(d)

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{\ln(1 + \sqrt{x-1})}{\sqrt{x^2-1}} &= \lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x+1}} \cdot \lim_{x \rightarrow 1^+} \frac{\ln(1 + \sqrt{x-1})}{\sqrt{x-1}} = \\ &= \frac{1}{\sqrt{2}} \lim_{x \rightarrow 1^+} \ln(1 + \sqrt{x-1})^{\frac{1}{\sqrt{x-1}}} = \frac{1}{\sqrt{2}}, \end{aligned}$$

abbiamo sfruttato il fatto che $\lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x-1}} = +\infty$, quindi il limite notevole $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$.

(e) Poniamo $e^x - 1 = y$, quindi $x \rightarrow 0 \Leftrightarrow y \rightarrow 0$. Quindi $x = \ln(y+1)$, sostituendo si ha:

$$\lim_{y \rightarrow 0} \frac{y}{\ln(y+1)} = \lim_{y \rightarrow 0} \frac{1}{\ln(y+1)^{\frac{1}{y}}} = 1$$

(f)

$$\lim_{x \rightarrow 0^+} (1 + |\sin x|)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{\frac{1}{|\sin x|}} \right)^{\frac{1}{|\sin x|} \cdot \frac{|\sin x|}{x}} = e^1 = e.$$

(g)

$$\lim_{x \rightarrow \infty} x e^x \sin \left(e^{-x} \sin \frac{2}{x} \right) = \lim_{x \rightarrow \infty} \frac{\sin \left(e^{-x} \sin \frac{2}{x} \right)}{e^{-x} \sin \frac{2}{x}} \cdot \lim_{x \rightarrow \infty} e^{-x} \sin \frac{2}{x} \cdot e^x \cdot x,$$

tenendo conto che:

$$e^{-x} \sin \frac{2}{x} \rightarrow 0,$$

$$x \sin \frac{2}{x} = 2 \frac{x}{2} \sin \frac{2}{x} = 2 \frac{\sin \frac{2}{x}}{\frac{2}{x}},$$

il limite cercato è 2.

(h)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\sin^2 3x} &= \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{(2x)^2} \cdot \lim_{x \rightarrow 0} \frac{4x^2}{\sin^2 3x} = \\ &= \frac{1}{2} \cdot 4 \cdot \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x} \right)^2 \cdot \frac{1}{9} = \frac{2}{9}. \end{aligned}$$

(i) Osserviamo innanzitutto che se $x \rightarrow 0$, $\pi \cos x \rightarrow \pi$, e che $0 < \pi \cos x < \pi$ ($\sin x = \sin(\pi - x) \forall x \in (0, \pi)$).

Quindi:

$$\lim_{x \rightarrow 0} \frac{\sin(\pi \cos x)}{x \sin x} = \lim_{x \rightarrow 0} \frac{\sin(\pi - \pi \cos x)}{\pi - \pi \cos x} \cdot \lim_{x \rightarrow 0} \pi \frac{1 - \cos x}{x^2} \cdot \frac{x}{\sin x} = 1 \cdot \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}.$$

$$(j) \lim_{x \rightarrow \frac{\pi}{2}} \tan x (e^{\cos x} - 1) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\cos x} \sin x = 1.$$

ESERCIZIO 2

(a)

$$\lim_{x \rightarrow 0^-} \frac{x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x}{-x} = -1,$$

$$\lim_{x \rightarrow 0^+} \frac{x}{|x|} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1.$$

(b)

$$\lim_{x \rightarrow 4^-} [x]\{x\} = \lim_{x \rightarrow 4^-} 3\{x\} = 3,$$

$$\lim_{x \rightarrow 4^+} [x]\{x\} = \lim_{x \rightarrow 4^+} 4\{x\} = 0.$$

(c)

$$\lim_{x \rightarrow 0^-} \frac{x \cos x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x \cos x}{-x} = -1,$$

$$\lim_{x \rightarrow 0^+} \frac{x \cos x}{|x|} = \lim_{x \rightarrow 0^+} \frac{x \cos x}{x} = +1.$$

$$\lim_{x \rightarrow 0^-} |x|^{\frac{1}{x}} = \lim_{x \rightarrow 0^-} -x^{\frac{1}{x}} = 0,$$

$$\lim_{x \rightarrow 0^+} |x|^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} x^{\frac{1}{x}} = +\infty.$$